GEOMETRY OF ARITHMETIC DIFFERENCE OPERATORS

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Abstract. The field of differential algebraic geometry is created by expanding algebraic geometry to include algebraic differential equations and their arithmetic analogs. In particular, there are four classes of operators that can be used to "enlarge usual algebraic geometry" by "adjoining" an operator δ [3]. When the adjoined operator is an arithmetic difference operator, arithmetic difference geometry is the result. Much of the basic theory of arithmetic difference operators parallels the theory that occurs when the operator is an arithmetic analog of a derivation called a p-derivation. In this paper we detail the basic theory of arithmetic difference operators noting the many parallels to and some differences from the theory in the case of p-derivations.

1. Introduction

In [3], A. Buium describes four classes of operators that may be used to "enlarge usual algebraic geometry". Two of these operators are ideally suited for arithmetic purposes, specifically p-derivations, an arithmetic analog of a derivation, and π-difference operators, an arithmetic difference operator that in fact lies morally somewhere between a usual derivation and a p-derivation. Previously not much has been written about π-difference operators. While the existence of a geometry arising from the adjoining of a π-difference operators is discussed in [4] and [3], no details of the basic theory in the case of π-difference operators are given. In what follows we give an introduction to the basic theory of the geometry that arises from adjoining a π-difference operator along the same lines as found in [5] and [2] for the case when the adjoined operator is a p-derivation.

Unsurprisingly much of the geometry in the case when π-difference operator is adjoined is very similar to the case when the adjoined operator is a p-derivation. Both the construction of π-jet spaces and the definitions of δ-formal functions and δ-characters are analogous in the case of π-difference operators to the case of p-derivations. What is remarkable is the fact that for π-difference operators, the group of δ-characters for an abelian group scheme G is a finitely generated torsion free R-module.

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In what follows we first introduce π-difference operators and then detail the construction of π-jet spaces using π-difference operators. This construction is central for the geometries arising from adjoining not just π-difference operators, but also p-derivations and usual derivations. After the construction of π-jet spaces, we discuss δ-formal functions and δ-characters. Finally we use p-difference operators to provide a quantitative bound for $\#(X(R) \cap J(R)_{tors}$ where X/R is a smooth projective curve of genus $g \geq 2$ and J/R is the Jacobian of X. This was done quite successfully using p-derivations. While it is also possible to provide a quantitative bound using π-difference operators, the results provide a caution that in the future the best applications of π-difference operators will lie in different areas than the best applications to date of p-derivations.

Let K be a field of characteristic zero, complete under discrete valuation ν, with an algebraically closed residue field k of characteristic $p > 0$. Furthermore assume K is a finite separable extension of $\mathbb{Q}_p$ with R the valuation ring of K and $e = \nu(p)$ the absolute ramification index. Fix $\pi \in R$ a prime element. For R-algebras A and B with $f : A \to B$ an R-algebra homomorphsim, a π-difference operator of f is a map $\delta : A \to B$ that satisfies

$$\begin{aligned} \delta(x+y) &= \delta x + \delta y \\ \delta(xy) &= f(y)\delta x + f(x)\delta y + \pi \delta x \delta y \\ \delta(1) &= 0 \end{aligned}$$

for all $x, y \in A$. Anytime it is clear that f is either the identity or a natural embedding of A into B we replace $f(x)$ with x.

A simple calculation shows that when $A = R$, the map $\theta : R \to B$ defined by $\theta(x) = x + \pi\delta(x)$ is a ring homomorphism. Furthermore θ is injective, and if $B = R$, then θ is an automorphism fixing $\mathbb{Z}_p$. Let $\mathcal{G}_0 \subset Gal(K/\mathbb{Q}_p)$ denote the subset of Galois automorphisms of K over $\mathbb{Q}_p$ that operate trivially on k, namely the first inertia group.

Proposition 1.1. *The set of π-difference operators from R to R is in bijective correspondence with the elements of the first inertia group $\mathcal{G}_0$.*

Proof. The hardest part of this is to show $\phi(x) = x + \pi\delta(x)$ is injective if δ is a π-difference operator. First realize that δ takes any element in $\mathbb{Z}_p$ to zero and therefore ϕ fixes $\mathbb{Z}_p$. Next we note that any non-trivial ideal in R contains non-zero elements of $\mathbb{Z}_p$. Therefore $\ker \phi$ can't be a non-trivial ideal. Surjectivity follows from the fact that ϕ must take the roots of any irreducible polynomial $f(x) \in \mathbb{Z}_p[x]$ to the roots of $f(x)$ bijectively. Also it is automatic that ϕ operates trivially on k. Finally it is a matter of simple computation to check that for each element $\phi \in \mathcal{G}_0$, the map $\delta(x) = \frac{\phi(x)-x}{\pi}$ is the corresponding π-difference operator. □

From now on fix $\phi \in \mathcal{G}_0$ and let δ be the corresponding π-difference operator from R to R defined by

$$\delta(x) = \frac{\phi(x) - x}{\pi}$$

for all $x \in R$. If ϕ is the identity element of the inertia group, then δ is the trivial, namely the zero map.

2. Jet spaces with π-difference operators

In an analogous fashion to the construction of jet spaces found in [5] and [2] we can construct jet spaces using π-difference operators. Even though the construction is very similar to the construction of jet spaces using p-derivations, we provide a complete overview, albeit brief. From now on we will refer to jet spaces constructed with π-difference operators as π-jet spaces to distinguish them from jet spaces constructed with derivations or p-derivations.

It will be necessary to use formal schemes in what follows. By a π-formal scheme we mean a Noetherian formal scheme X/R such that $\pi\mathcal{O}_X$ is the ideal of definition of X. If X/R is a scheme, then we will denote its associated formal scheme by $\hat{X}/R$. If S is a π-adically complete ring, we let $Spf\ S$ be the formal scheme obtained by completing $\operatorname{Spec} S$ along the closed subscheme defined by the ideal πS. In turn, a π-formal group scheme is a group object in the category of π-formal schemes. With this convention, if G/R is a group scheme, then $\hat{G}/R$ is the associated π-formal group scheme. For G/R a group scheme, set $G(\pi R) := \ker(G(R) \to G_0(k))$ where $G(R) := \operatorname{Hom}(\operatorname{Spec} R, G)$ and $G_0 = G \otimes k$. Then the set $G(\pi R)$ is isomorphic to an g-dimensional formal group defined over R where g is the dimension of G. We will denote this formal group by $\mathcal{F}_G$. For more details about g-dimensional formal groups see [8].

We begin the construction of π-jet spaces by introducing the following universality property. Let

$$R \overset{f,\delta}{\to} B \overset{f^1,\delta}{\to} B^1 \overset{f^2,\delta}{\to} B^2 \to \ldots \to B^{n-1} \overset{f^n,\delta}{\to} B^n \to \ldots$$

be a sequence of R-algebras where each f^i is an algebra morphism and each δ is a π-difference operator of the appropriate f^i that extends the π-difference operator $\delta : R \to R$ such that $\delta \circ f^{n-1} = f^n \circ \delta$ with the following universality property:

Proposition 2.1 (UNIVERSALITY PROPERTY)**.** *Let $g : B^{n-1} \to C$ be any ring homomorphism of R-algebras and let ∂ be a p-difference operator from B^{n-1} to C of g such that $\partial \circ f^{n-1} = g \circ \delta$. Then there exists a unique ring homomorphism $u : B^n \to C$ such that $g = u \circ f^n$ and $\partial = u \circ \delta$.*

By the universality property the sequence

$$R \overset{f,\delta}{\to} B \overset{f^1,\delta}{\to} B^1 \overset{f^2,\delta}{\to} B^2 \to \ldots \to B^{n-1} \overset{f^n,\delta}{\to} B^n \to \ldots$$

satisfying the above properties for a given R-algebra B is unique up to isomorphism. We shall call such a sequence a *π-jet sequence*. It is simple to see that the sequence exists for a given R-algebra as demonstrated by the following construction. For $B = R[T]$ where T is a tuple of variables, let $B^n = R[T, T', T'', \dots, T^{(n)}]$ where each $T^{(i)}$ is a tuple of variables. In this case $f : R \to B$ would be the natural inclusion as would $f^n : B^{n-1} \to B^n$. Extend $\phi : R \to R$ to a homomorphism from B to B^1 by letting $\phi(T_i) = T_i + \pi T_i'$ for all T_i in T. Then $\delta : B \to B^1$ defined by $\delta(x) = (\phi(x) - x)/\pi$ is a π-difference operator of the natural inclusion that extends our fixed $\delta : R \to R$ and hence we will use the same letter for both. Inductively the extension of ϕ to $\phi : B^{n-1} \to B^n$ is completely defined by the behavior of ϕ on $T^{(n-1)}$. We define ϕ's behavior to be $\phi(T_i^{(n-1)}) = T_i^{(n-1)} + \pi T_i^{(n)}$. Since extending ϕ is equivalent to extending δ for $\delta(x) = (\phi(x) - x)/\pi$, this gives us a π-difference operator $\delta : B^{n-1} \to B^n$ of f^n. It is straightforward to check that this sequence satisfies the properties of a π-jet sequence.

We now consider the more general case of $B = R[T]/I$. Here we let $B^1 = R[T, T']/(I, I')$ where I' is the ideal generated by the image of I under δ under the above maps. We let $f : B \to B^1$ be the natural inclusion. In this setting, the extension of δ just constructed is a π-difference operator from B to B^1 extending $\delta : R \to R$. More generally we define $I^{(n)}$ to be the ideal generated by the image of $I^{(n-1)}$ under δ and let $B^n = R[T, T', \dots, T^{(n)}]/(I, I', \dots, I^{(n)})$ with $f^n : B^{n-1} \to B^n$ the natural inclusion. Then $\delta : R[T, T', \dots, T^{(n-1)}] \to R[T, T', \dots, T^{(n)}]$ as previously constructed is π-difference operator from B^{n-1} to B^n. Once again it is straightforward to check that this more general sequence satisfies the property that $\delta \circ f^{n-1} = f^n \circ \delta$ and the universality property.

In general these sequences do not behave well with respect to localization. While $(B^1)_t$ is a subring of $(B_t)^1$ for $t \in B$, these algebras are not necessarily equivalent unless either π is nilpotent or R is π-adically complete. However, for

$$R \xrightarrow{f,\delta} B \xrightarrow{f^1,\delta} B^1 \xrightarrow{f^2,\delta} B^2 \to \dots \to B^{n-1} \xrightarrow{f^n,\delta} B^n \to \dots$$

a π-jet sequence, we can consider the sequence

$$R \xrightarrow{\hat{f},\delta} \hat{B} \xrightarrow{\hat{f}^1,\delta} \hat{B}^1 \xrightarrow{\hat{f}^2,\delta} \hat{B}^2 \to \dots \to \hat{B}^{n-1} \xrightarrow{\hat{f}^n,\delta} \hat{B}^n \to \dots$$

where each algebra $\hat{B}^i$ is the π-adic completion of B^i, $\hat{f}^i$ is the extension of f^i to the π-adic completion, and δ is extended to the π-adic completions. Then this π-adically complete sequence satisfies the property that $\delta \circ \hat{f}^{n-1} = \hat{f}^n \circ \delta$ and a similar universality property in which the condition 'any R-algebra C' is changed to the condition 'any π-adically complete R-algebra C'. We call such a sequence the *π-jet formal sequence* and note that π-jet formal sequences behave well with respect to localization, namely $(\hat{B}^n)_t \simeq (\hat{B}_t)^n$ for $t \in B$.

The existence of π-jet formal sequences allow us to construct for any scheme of finite type X/R a sequence of π formal schemes

$$\ldots \to X^n \stackrel{f^n}{\to} X^{n-1} \to \ldots \to X^1 \stackrel{f}{\to} X^0 = \hat{X}$$

such that for any open affine subset $U \subset X$, the sequence

$$R \to \mathcal{O}_{X^0}(U) \to \mathcal{O}_{\hat{X}^1}(f^{-1}(U)) \to \ldots$$

is a π-jet formal sequence. We define the nth order π-jet space to be the π-formal scheme X^n. From now on we will apply on occasion a slight abuse of notation and use $\mathcal{O}_{\hat{X}^n}$ for both the structure sheaf of $\hat{X}^n$ and the appropriate direct image sheaf on $\hat{X}^i$ for $i < n$. Furthermore when the context is clear we will drop the subscripts completely and denote the appropriate structure sheaf by $\mathcal{O}$. We define X^∞ to be the π-adic completion of the inverse limit of the π-jet spaces, namely $X^\infty = \operatorname{inv\,lim} X^n$. Note that in the affine case where $X = \operatorname{Spec} B$ for $B = R[T]/I$, the formal scheme X^∞ is simply $Spf\ B^\infty$ where $B^\infty = R[T, T', T'', \ldots, T^{(n)}, \ldots]/(I, I', \ldots, I^{(n)}, \ldots)$. We define the k-scheme X_0^n to be $X^n \otimes k$ for any integer n and in turn define the scheme X_0^∞ as the k-scheme $\operatorname{inv\,lim} X_0^n$.

The universality property allows us to construct maps

$$\nabla^n : X(R) \to X^n(R)$$

where $X(R) = \operatorname{Hom}(\operatorname{Spec} R, X)$ and $X^n(R) = \operatorname{Hom}(Spf\ R, X^n)$. For any $P \in X(R)$, there is a π-jet formal sequence of the ring of regular functions in a neighborhood of P. We note that δ extends to a π-difference operator from the ring of regular functions in a neighborhood of P to R. Then by inductively applying the universality property to the π-jet formal sequence of the ring of regular functions we get an R valued point in X^n. This construction extends to $\nabla^\infty : X(R) \to X^\infty(R)$, the projective limit of these maps. The composition of ∇^n with the canonical reduction map from R to k gives us a map

$$\nabla_0^n : X(R) \to X^n(R) \to X_0^n(k).$$

If we pass to the projective limit of the maps ∇_0^n we have $\nabla_0^\infty : X(R) \to X_0^\infty(k)$. In the case of $X = \mathbf{A}^1$, the affine line, the nth order π-jet space is $X^n = Spf\ R[T, T', \ldots T^{(n)}]$, ∇^∞ can be described as a map from $R \to R^{\mathbb{N}}$ that takes

$$a \mapsto (a, \delta a = a', \delta^2 a = a'', a''', \ldots)$$

and if $pr : R \to k$ is the canonical projection map, then ∇_0^∞ takes

$$a \mapsto (pr(a), pr(a'), pr(a''), pr(a'''), \ldots).$$

Proposition 2.2. *Let m be the order of ϕ, the smallest integer such that $\phi^m(x) = x$. Suppose $X = \mathbf{A}^1$. Then there exist linear polynomials $P_i \in R[x_0, ..., x_{m-1}]$ such the map ∇^∞ takes*

$$a \mapsto (a, a', a'', \ldots, a^{(m-1)}, P_1(a, a', a'', \ldots, a^{(m-1)}), P_2(a, a', a'', \ldots, a^{(m-1)}), \ldots).$$

Proof. Let $x \in R$. Then

$$\phi(x) = x + \pi x',$$
$$\phi^2(x) = x + (\pi + \phi(\pi))x' + (\pi\phi(\pi))x'',$$

and by induction

$$\phi^n(x) = x + S_{1n}x' + S_{2n}x'' + \ldots + S_{nn}x^{(n)}$$

where S_{jk} is the symmetric polynomial on $\pi, \phi(\pi), \ldots, \phi^{k-1}(\pi)$ with each term in S_{jk} of degree j and such that S_{jk} is of degree one in $\phi^i(\pi)$ for $0 \leq i < k$. So

$$\phi^m(x) = x = x + S_{1m}x' + S_{2m}x'' + \ldots + S_{mm}x^{(m)}$$

meaning $S_{1m}x' + S_{2m}x'' + \ldots + S_{mm}x^{(m)} = 0$. Furthermore $\phi(S_{im}) = S_{im}$ and so applying δ to this equation we have

$$S_{1m}x^{(i+1)} + S_{2m}x^{(i+2)} + \ldots + S_{mm}x^{(i+m)} = 0$$

for all $i \geq 0$. It is also important to note that S_{mm} divides S_{im} for $1 \leq i \leq m$.

We can now construct the P_i inductively. Let

$$P_1(x_0, ..., x_{m-1}) = \left(\frac{-1}{S_{mm}}\right)\left(S_{1m}x_1 + S_{2m}x_2 + \ldots + S_{m-1,m}x_{m-1}\right).$$

Then let

$$P_2(x_0, ..., x_{m-1}) = \left(\frac{-1}{S_{mm}}\right)\left(S_{1m}x_2 + \ldots + S_{m-2,m}x_{m-1} + S_{m-1,m}P_1(x_0, ..., x_{m-1})\right)$$

and

$$P_3(x_0, ..., x_{m-1}) = \left(\frac{-1}{S_{mm}}\right)\left(S_{1m}x_3 + \ldots + S_{m-2,m}P_1(x_0, ..., x_{m-1}) + S_{m-1,m}P_2(x_0, ..., x_{m-1})\right).$$

In general using the fact that

$$S_{mm}x^{(m+i)} = -\left(S_{1m}x^{(i+1)} + S_{2m}x^{(i+2)} + \ldots + S_{m-1,m}x^{(i+m-1)}\right),$$

we can continue in this fashion to construct polynomials P_i such that

$$P_i(x, x', x'', \ldots, x^{(m-1)}) = x^{(m+i-1)}.$$

Since substituting a linear polynomial in for a linear term gives a linear polynomial, all of the P_is are linear. □

From now on S_{jk} will be the symmetric polynomial on $\pi, \phi(\pi), \ldots, \phi^{k-1}(\pi)$ with each term in S_{jk} of degree j and such that S_{jk} is of degree one in $\phi^i(\pi)$ for $0 \leq i < k$.

Corollary 2.3. *Let m be the order of ϕ, the smallest integer such that $\phi^m(x) = x$. Suppose $X = \mathbf{A}^n$. Then there exist linear polynomials $P_i \in R[x_0, ..., x_{m-1}]$ such the map ∇^∞ takes*

$$a \mapsto (a, a', a'', \ldots, a^{(m-1)}, P_1(a, a', a'', \ldots, a^{(m-1)}), P_2(a, a', a'', \ldots, a^{(m-1)}), \ldots)$$

where the P_i are regarded as linear transformations taking n-tuples to n-tuples.

Remark 2.4. Note that unlike with p-derivations, the map $\nabla_0^\infty : X(R) \to X_0^\infty(k)$ is distinctly not bijective.

The π-jet space construction has a local product property like p-jet spaces. To see this we note first the following proposition and lemma that mirror similar statements found in [2] and [4]. Both the proposition and lemma can be proved using the same techniques found in [2] and also present in [4]. For this reason only the proof of the lemma will even be mentioned since it requires a very minor adjustment.

Lemma 2.5. *Let A be an R-algebra, $u : A \to B$ a finitely generated étale A-algebra, and $v : B \to C$ a ring homomorphism into a π-adically complete ring C. Then any π-difference operator of $v \circ u$ lifts uniquely to a π-difference operator of v.*

Proof. This is easily done using an identical proof structure to the proof found in [2]. The ring $W_2^\pi(C)$ is replace by the ring $V_2(C) \simeq C[t]/(t^2 - \pi t)$ and then the argument in [2] applies. □

Proposition 2.6. *Let $u : R[x] \to B$ be an étale morphism of finite type where x is a finite family of indeterminates. Let $x', x'', \ldots, x^{(r)}$ be families of new indeterminates indexed by the same set as x. Then the natural morphism*

$$B[x', x'', \ldots, x^{(r)}] \to \hat{B}^n$$

that sends $x^{(i)}$ to $\delta^i(u(x))$ is an isomorphism.

From these we conclude the local product property.

Corollary 2.7. *(Local Product Property) Suppose X/R is smooth scheme of finite type and relative dimension g. Then each point in X has an open neighborhood U such that the π-jet spaces of U have the product decomposition*

$$U^n \simeq \hat{U} \hat{\times} \hat{\mathbf{A}}^{gn}$$

as π-formal schemes.

Further geometric properties of π-jet spaces are as follows. In the cases where the property is analogous to the same property for p-jet spaces and the property can be proved with at most minor changes to the proof given for p-jet spaces, we refer the reader to the appropriate reference. The following technical lemma while not completely analogous, is necessary to the proof the proposition immediately following.

Lemma 2.8. *Suppose T is a tuple of variables. For $F \in R[T, T', \dots, T^{(n)}]$*

$$\phi\left(\frac{\partial F}{\partial T_i^{(n)}}\right) = \frac{\partial F'}{\partial T_i^{(n+1)}}.$$

Proof. It is enough to check $F = ax^m$ where $x = T_i^{(n)}$ and the $\gcd(a, x) = 1$ since π-difference operators are additive. Computing the left side of the formula,

$$\phi\left(\frac{\partial F}{\partial T_i^{(n)}}\right) = \phi\left(\frac{\partial F}{\partial x}\right) = \phi\left(amx^{m-1}\right) = \phi(a)\phi(m)\phi(x)^{m-1} = \phi(a)m(x+\pi x')^{m-1}.$$

Applying δ to F, we have $F' = x^m\delta(a) + \delta(x^m)(a + \pi\delta a)$ or

$$F' = x^m\delta a + \phi(a)\left(\frac{(x+\pi x')^m - x^m}{\pi}\right).$$

So

$$\frac{\partial F'}{\partial x'} = \frac{\phi(a)}{\pi}(x+\pi x')^{m-1}m\pi = \phi\left(\frac{\partial F}{\partial T_i^{(n)}}\right).$$

□

With this technical lemma, the following proposition can be proved in the same fashion as Proposition 2.2 in [5].

Proposition 2.9. *Suppose X/R is smooth along X_0. Then the morphisms $X^n \to X^{n-1}$ are smooth in the sense that they are locally obtained as π-adic completions of smooth morphisms of schemes.*

With 2.9, the following lemma follows directly from a similar proof to that given in [5] for Lemma 1.6 with $W^2(B)$ replaced by $V^2(B) \simeq B[t]/(t^2 - \pi t)$. While this is certainly in some sense just a variant of lemma 2.5, we include it here for parallelism purposes so that the proof for proposition 2.11 follows [5].

Lemma 2.10. *Suppose X/R is smooth along X_0. Let U be an open affine subset of X and let $B = \mathcal{O}_X(U)$ and $\hat{f}^n : \hat{B}^{n-1} \to \hat{B}^n$ part of the π-jet formal sequence. Then for any $m \geq 1$, the natural projection $pr : \hat{B}^{n-1} \to B^{n-1}/\pi^m$ factors through $\hat{f}^n$.*

This in turn is sufficient for the following proposition. First we recall that for X/R a smooth scheme of finite type over R, the tangent bundle of X_0 denoted by $TX_0 \to X_0$ is simply $\mathrm{Spec}\,(S(\Omega_{X_0/k}))$. More generally the relative tangent bundle of $X_0^{n-1} \to X_0^{n-2}$ is $T(X_0^{n-1}/X_0^{n-2}) := \mathrm{Spec}\,(S(\Omega_{X_0^{n-1}/X_0^{n-2}})) \to X_0^{n-1}$.

Proposition 2.11. *Suppose X/R is smooth along X_0. Then for $n \geq 1$, $X_0^n \to X_0^{n-1}$ is a (Zariski locally trivial) principal homogenous space of the relative tangent bundle $T(X_0^{n-1}/X_0^{n-2}) \to X_0^{n-1}$. In addition if X/R is a group scheme then $\ker(X_0^n \to X_0^{n-1})$ is a vector group for all positive integers n.*

In the case of $n = 1$, this proposition means that $X_0^1 \to X_0$ is a (Zariski locally trivial) principal homogeneous space for the tangent bundle of X_0 denoted by $TX_0 \to X_0$. As such it has an associated cohomology class in $H^1(X_0, TX_0)$. In particular this cohomology class is computed explicitly as follows for X/R. Let $\{U_i\}$ be an open affine cover of X_0 and for each i, let δ_i be the π-difference operator lifting δ on U_i. The cohomology class is the collection of differences $(\delta_i - \delta_j)$ modulo π for $i < j$ which is a class in $\check{H}^1(X_0, TX_0)$. Note that modulo π, π-difference operators are derivations. It is important to note that this cohomology class coincides exactly with the Kodaira-Spencer class encountered in classical deformation theory. Therefore saying X/R has a non-trivial Kodaira-Spencer class is equivalent to saying that the cohomology class in $H^1(X_0, TX_0)$ associated to the principal homogenous space $X_0^1 \to X_0$ for $TX_0 \to X_0$ is non-trivial. In the particular case of curves, this means the following.

Proposition 2.12. *Let X/R be a smooth projective curve whose Kodaira-Spencer class is non-trivial. Then X_0^n is affine for $n \geq 1$.*

Proof. First consider the case $n = 1$. The cohomology class in $H^1(X_0, TX_0)$ associated to the principal homogenous space $X_0^1 \to X_0$ for $TX_0 \to X_0$ is non-zero. At this point the proof of Proposition 1.10 [5] applies mutatis mutandi. Therefore X_0^1 is affine. However, if X_0^1 is affine, if follows by induction that X_0^n is affine for $n \geq 1$. □

3. δ-formal functions and δ-characters for δ a π-difference operator

The definition of δ-formal functions in the case of π-difference operators is analogous to the same definition for p-derivations. Namely

Definition 3.1. Let X/R be a scheme of finite type and $\delta : R \to R$ a fixed π-difference operator. An R-valued function $\varphi : X(R) \to R$ will be called a δ-formal function of order $\leq r$ on $X(R)$ if for any point in $X(R)$ there is an open affine neighborhood $U \subset X$ and a closed embedding $t : U \to \mathbf{A}^N$ with $N \geq 1$ such that φ can be written as

$$\varphi(P) = \Phi\big(t(P), t(P)', \ldots, t(P)^{(r)}\big)$$

for $P \in U(R)$ and Φ an element in $R[x, x', \ldots x^{(r)}]\hat{}$ with each $x^{(i)}$ an N-tuple of indeterminates.

Let $\mathcal{O}^r(X)$ be the ring of all δ-formal functions of order $\le r$ on $X(R)$ and let $\mathcal{O}^\infty(X)$ be the ring formed by the set of all δ-formal functions. There is a natural map $\mathcal{O}(X^r) \to \mathcal{O}^r(X)$ and hence $\mathcal{O}(X^\infty) \to \mathcal{O}^\infty(X)$. Explicitly, let $f \in \mathcal{O}(X^r)$. Then for each $P \in X(R)$, $\nabla(P) \in X^r(R) = \mathrm{Hom}(Spf\ R, X^n)$ has a corresponding ring homomorphism $\mathcal{O}(X^r) \to R$ that takes f to some value in R. The δ-formal function $\hat{f} : X(R) \to R$ is the function that takes P to this value. So for example in the case when X/R is affine $\hat{f}(P) = f(\nabla(P))$. By its construction, this map is automatically surjective. However, for $r \ge m$ where m is the smallest integer such that ϕ^m is the identity function, this map is never injective. In fact as we shall see all δ-formal functions are equivalent to a δ-formal function of order less than m.

Lemma 3.2. *Let m be the smallest integer such that ϕ^m is the identity function. Then $\mathcal{O}^\infty(X)$ is the image of $\mathcal{O}(X^{m-1})$ under the natural map $\mathcal{O}(X^{m-1}) \to \mathcal{O}^{m-1}(X)$. Namely any δ-formal function is of order $\le m-1$.*

Proof. Since this is an affine question, we may assume that $X = \mathrm{Spec}\, R[T]/I$ for T and N-tuple and that $Y = \mathbf{A}^N$. Let $r \ge m$. Define $\psi^\# : R[T, \ldots, T^{(r)}] \to R[T, \ldots, T^{(m-1)}]$ to be the map given by $T^{(i)} \mapsto T^{(i)}$ for $i \le m-1$ and $T^{(i)} \mapsto P_{i-m+1}(T, T', T'', \ldots, T^{(m-1)})$ for $i > m-1$ where P_{i-m+1} are the linear polynomials in corollary 2.3. Then $\psi^\#$ defines a map $\psi : Y^{m-1} \to Y^r$ such that $\nabla^r(P) = \psi(\nabla^{m-1}(P))$ for all $P \in Y(R)$.

Let $\varphi : X(R) \to R$ be a δ-formal function of degree r. Then there exists $f \in \mathcal{O}^r(X)$ such that $\hat{f} = \varphi$. Let $t : X \to Y$ be the natural closed embedding which lifts to a closed embedding of $X^r \to Y^r$. Let $F \in \mathcal{O}(Y^r)$ be in the preimage of f under the ring map $\mathcal{O}(Y^r) \to \mathcal{O}(X^r)$ associated to the closed embedding. Then for $P \in X(R)$,

$$\varphi(P) = \hat{f}(P) = F(\nabla^r(t(P))) = F(\psi \circ \nabla^{m-1}(t(P))) = \psi^\#(F(\nabla^{m-1}(t(P)))).$$

Let $g = \psi^\#(F)$. Then $g \in \mathcal{O}(Y^{(m-1)})$ and $\varphi(P) = g(\nabla^{m-1}(t(P)))$ meaning the order of φ is $\le m-1$. Finally if g^* is the image of g under the ring map $\mathcal{O}(Y^{m-1}) \to \mathcal{O}(X^{m-1})$ associated to the closed embedding, then $\varphi = \hat{g}^*$. □

Corollary 3.3. *If ϕ is the identity function, then $\mathcal{O}^\infty(X) \simeq \mathcal{O}(\hat{X})$.*

For the rest of this section we assume G is a smooth commutative group scheme of finite type over R. Note that as in the case of jet spaces constructed with p-derivations, the nth π-jet space of a group scheme G/R is a π-formal group scheme G^n/R. We can now define additive δ-characters for $G(R)$.

Definition 3.4. Suppose G/R is a smooth commutative group scheme of finite type and δ a fixed π-difference operator. A δ-character for $G(R)$ is a δ-formal function $\psi : G(R) \to R = \mathbb{G}_a(R)$ which is also an additive group homomorphism where $\mathbb{G}_a$ is the one-dimensional additive group over R.

A δ-character has an order $\leq r$ if the δ-formal function has order $\leq r$. Let $\mathbb{X}^r(G)$ be the group of all δ-characters of order $\leq r$ and let $\mathbb{X}^\infty(G)$ be the group of all δ-characters. Then by lemma 3.2, all δ-characters are of order $\leq m-1$ where m is the smallest integer such that ϕ^m is the identity function. Also we can identify $\mathbb{X}^r(G)$ with $\mathrm{Hom}(G^n, \hat{\mathbb{G}_a})$ where Hom refers to homomorphisms of group schemes. Using this identification we first describe δ-characters for $\mathbb{G}_a$. Then we consider δ-characters for G/R.

For any G/R a smooth scheme of finite type and relative dimension g fix a point $\rho \in G(R)$, $\rho : \mathrm{Spec}\, R \to G$. Let $P, Q \in G$ be the image under ρ of the generic point and closed point respectively. Let x be a regular system of parameters of $\mathcal{O}_{G,P}$ which is contained in $\mathcal{O}_{G,Q}$. If $\mathfrak{q}$ is the ideal of Q, then the g-tuple x provides inclusions $R[x] \subset \mathcal{O}_{G,Q} \subset R[[x]]$ where $R[[x]]$ is identified with the completion of $\mathcal{O}_{G,Q}$ along $\mathfrak{q}$. Denote also by Q the image of the closed point under $\nabla^n \rho : Spf\ R \to G^n$. Then the $\mathfrak{q}$-completion of $\mathcal{O}_{G^n,Q}$ is identified with $R[[x, x', \ldots, x^{(n)}]]$ by the local product property. Furthermore if the closed fiber G_0/k is connected, the map $\mathcal{O}^n(G) \to R[[x, x', \ldots, x^{(n)}]]$ is injective.

Proposition 3.5. *Let m be the smallest integer such that ϕ^m is the identity function on R. Then $\mathbb{X}^\infty(\mathbb{G}_a)$ is a finitely generated free R-module with basis $\{\phi^0, \phi^1, \ldots, \phi^{m-1}\}$.*

Proof. Let $\psi \in \mathbb{X}^\infty(\mathbb{G}_a)$. Then by lemma 3.2, $\psi \in \mathbb{X}^{m-1}(\mathbb{G}_a)$ and so we can identify ψ with an element also denoted by $\psi \in R[[x, x', \ldots, x^{(m-1)}]]$. Consider the K-algebra isomorphism $\sigma : K[[y_0, \ldots, y_{m-1}]] \to R[[x, x', \ldots, x^{(m-1)}]]$ defined by

$$\begin{aligned}
\sigma(y_0) &= \frac{x}{S_{m-1,m-1}} \\
\sigma(y_1) &= \frac{\phi(x)}{S_{m-1,m-1}} = \frac{1}{S_{m-1,m-1}}\left(x + S_{11}x'\right) \\
\sigma(y_2) &= \frac{\phi^2(x)}{S_{m-1,m-1}} = \frac{1}{S_{m-1,m-1}}\left(x + S_{12}x' + S_{22}x''\right) \\
&\vdots \\
\sigma(y_{m-1}) &= \frac{\phi^{m-1}(x)}{S_{m-1,m-1}} = \frac{1}{S_{m-1,m-1}}\left(x + S_{1,m-1}x' + \ldots + S_{m-1,m-1}x^{(m-1)}\right)
\end{aligned}$$

It follows that $\sigma^{-1}(\psi)$ is additive in $y_0, \ldots, y_{m-1}$ and therefore $\psi = \sum_{i=0}^{m-1} \lambda_i \phi^i$ for $\lambda_i \in K$. Then the coefficient of x is $\frac{1}{S_{m-1,m-1}} \sum_{i=0}^{m-1} \lambda_i$, the coefficient of x' is $\frac{1}{S_{m-1,m-1}} \sum_{i=1}^{m-1} S_{1i}\lambda_i$, the coefficient of x'' is $\frac{1}{S_{m-1,m-1}} \sum_{i=2}^{m-1} S_{2i}\lambda_i$, etc. Starting with the coefficient of x^{m-1} which is in R and is λ_{m-1} we have $\lambda_{m-1} \in R$. Next

$\phi^{m-2}(\pi)$ times the coefficient of x^{m-2} is in R and is $\lambda_{m-2}+\frac{\phi^{m-2}(\pi)S_{m-2,m-1}}{S_{m-1,m-1}}\lambda_{m-1}$ meaning $\lambda_{m-2} \in R$. Proceeding in a similar fashion it follows that $\lambda_i \in R$ for all $0 \leq i \leq m-1$. Therefore ψ is a linear combination of $\{\phi^0, \phi^1, \ldots, \phi^{m-1}\}$ over R. By the linear independence of Galois automorphisms, all such linear combinations are non-trivial δ-characters. □

Remarkably we can now use much of the machinery found in Section 2 of [2] to show that for any commutative G/R, the set of δ-characters for G/R is a finitely generated R-module. To use this machinery let $\mathcal{F}$ be a g-dimensional formal group defined over R. Let $F(x_1, x_2) \in R[[x_1, x_2]]^g$ be the g-dimensional formal group law of $\mathcal{F}$ where x_1 and x_2 are g-tuples. Then $\pi^{-n}F(\pi^n x_1, \pi^n x_2)$ is the formal group law of a g-dimensional π-formal group on $\hat{\mathbf{A}}^g$ which we will denote by $\mathcal{F}\{n\}$ and is referred to as a *twist* of $\mathcal{F}$. The map of $\mathcal{F}$ to $\mathcal{F}\{n\}$ is a functor from formal group laws to π-formal groups and so if $\mathcal{F}_1$ and $\mathcal{F}_2$ define the same local formal group, then $\mathcal{F}_1\{n\}$ and $\mathcal{F}_2\{n\}$ are isomorphic π-formal groups. The proof of the following proposition follows the proof of Proposition 2.2 in [2].

Proposition 3.6. *Let G/R be a smooth group scheme of finite type and let $\mathcal{F}$ be any formal group associated to the local formal group $\mathcal{F}_G$. Let $F[x_1, x_2]$ be the g-dimensional group law of $\mathcal{F}$ and let $F^{\phi^n}[x_1, x_2]$ be the formal group law obtained by applying ϕ^n to the coefficients of F. Denote by $\mathcal{F}^{\phi^n}$ the formal group associated to $F^{\phi^n}[x_1, x_2]$. Then the kernel of $G^n \to G^{n-1}$ is isomorphic as a π-formal group to the twist $\mathcal{F}^{\phi^n}\{n\}$ of $\mathcal{F}^{\phi^n}$.*

Proof. This is easily done using an identical proof structure to that of Propsition 2.2 in [2] because of the many similarities between π-difference operators and p-derivations. It is necessary to replace p-derivations with π-difference operators and that $\partial_c : R[x, x', \ldots, x^{(n-1)}] \to C$ is defined by the formula

$$\partial_c f = (\delta h)(0, \ldots, 0, c) = \frac{1}{\pi}\big(h^{\phi}(0, \ldots, 0, \pi c) - h(0, \ldots, 0, 0)\big)$$

for $h \in R[x, x', \ldots, x^{(n-1)}]$ where C is as in the proof of Proposition 2.2 and h^{ϕ} is the polynomial obtained by applying ϕ to the coefficients of h. □

Next just as in section 2.7 of [2], if G/R is a smooth commutative group scheme of finite type of relative dimension g, we have a morphism of π-formal groups $u : \hat{\mathbb{G}_a}^g \to \hat{G}$ such that the induced map $u : R^g \to G(R)$ is injective. With this map we can prove the following theorem.

Theorem 3.7. *Let G/R be a smooth commutative group scheme of finite type and relative dimension g. Then $\mathbb{X}^{\infty}(G)$ is a finitely generated torsion free R-module of rank $\leq mg$ where m is the smallest integer such that ϕ^m is the identity function on R.*

Proof. The morphism u induces morphisms of π-formal groups between π-jet spaces $(\mathbb{G}_a^g)^n \to G^n$. If we apply the functor $\mathrm{Hom}(-, \hat{\mathbb{G}}_a)$ to these morphisms we get morphisms $u^* : \mathbb{X}^n(G) \to \mathbb{X}^n(\mathbb{G}_a^g)$ which are injective because $\psi \in \mathbb{X}^\infty(G)$ vanishes on $G(\pi^i R)$ only if $\psi = 0$. In particular $u^*(\mathbb{X}^\infty(G))$ is a submodule of $\mathbb{X}^\infty(\mathbb{G}_a^g)$ which by proposition 3.5 is a finitely generated free R-module of rank mg. □

4

An obvious application of arithmetic difference geometry is to provide quantitative bounds for $\#(X(R) \cap J(R)_{tors}$ where X/R is a smooth projective curve of genus $g \geq 2$ and J/R is the Jacobian of X. This has been done quite successfully using p-derivations and similar techniques work with π-difference operators. There are some key differences in the results though. With π-difference operators, it is necessary that X/R have a non-trivial Kodaira-Spencer class and the resulting bound is

$$\#(X(R) \cap J(R)_{tors}) \leq p^{(3+N)g}(3^g)[8g-2]g!$$

where N is the smallest integer such that $\frac{\nu(p)}{p^{N+1}-p^N} < 1$. The requirement that X/R have a non-trivial Kodaira-Spencer class means that $\nu(p) > 1$. However, a special case of a theorem of Coleman is that if X is a curve of genus g over $\mathbb{Q}_p$ with good reduction and $p > 2g$, then $X \cap J_{tors}$ is unramified, i.e., contained in $J(\mathbb{Q}_p{}^{unr})_{tors}$ [10],[7]. As a consequence even though this bound is an improvement over the bound attained using p-derivations for small values of e, for $p > 2g$ it cannot be used to bound $X(\mathbb{Q}_p) \cap J_{tors}(\mathbb{Q}_p)$.

For any smooth projective curve X/R we will use the notation J/R to refer to the Jacobian of X. The letter Γ will refer to the set of torsion points, $J(R)_{tors}$. Our convention will be to write J additively, and so by pJ we simply mean the image under the map that adds together p occurrences of a point in J. The following proposition is proved as a claim in the proof of Theorem 1.11 in [5] and then improved in a remark in the introduction of [1].

Proposition 4.1. *Let J be the Jacobian of a smooth projective curve X/R of genus g at least two. Let $\Gamma = J(R)_{tors}$. Then the number of points in $\Gamma/p\Gamma$ is at most p^g.*

Proof. On p. 356 of [5], Buium proves the claim that the number of points in $\Gamma/p\Gamma$ is at most p^{2g}. In a note on p. 4534 of [1], Boxall and Grant observe that in fact the number of points in $\Gamma/p\Gamma$ is at most p^g because of the Weil pairing. □

The next proposition is a well known fact about formal groups. For the convenience of the reader we sketch a proof.

Proposition 4.2. *Let R be a complete discrete valuation ring whose maximal ideal is generated by π with valuation ν and residue field of characteristic p. Let $\mathcal{F}$ be a g-dimensional formal group defined over R. If $x \in \mathcal{F}(\pi R)$ is non-zero element such that $[p^n](x) = 0$ and $[p^{n-1}](x) \neq 0$ namely an element of order p^n, then*

$$\nu(x_j) \leq \frac{\nu(p)}{p^n - p^{n-1}}$$

where j is the index between one and g such that $\nu(x_j) = \min\{\nu(x_i)|i = 1, \ldots g\}$.

Proof. Let $F(X,Y)$ be the g-dimensional formal group law of $\mathcal{F}$ and let $[l] : \mathcal{F} \to \mathcal{F}$ be the inductively defined homomorphism of formal groups for k an integer. Then

$$[p](T) = \big(pf_1(T) + h_1(T_1^p, \ldots, T_g^p), \ldots, pf_g(T) + h_g(T_1^p, \ldots, T_g^p)\big)$$

where f_i and h_i are power series in g-variables and $f_i(0) = 0, h_i(0) = 0$. We use the notation $[l]_i$ to denote the ith component of the homomorphism $[l]$ and the notation T^k to mean $(T_1^k, \ldots, T_g^k)$. Then

$$[p^k]_i(T) = p^k \bar{h}_{i,k}(T) + p^{k-1}\bar{h}_{i,k-1}(T^p) + \ldots + \bar{h}_{i,0}(T^{p^k})$$

where each of the $\bar{h}_{i,j}$ are power series in g variables such that $\bar{h}_{i,j}(0) = 0$.

Let $x \in \mathcal{F}(\pi R)$ be of order p^n. Fix j to be the index between one and g such that $\nu(x_j) = \min\{\nu(x_i)|i = 1, \ldots g\}$. Then $\nu(x_j) \leq \nu(p)/(p-1)$, so $\nu([p]_i(x)) \geq p\nu(x_j)$ and $\nu([p^k]_i(x)) \geq p^k \nu(x_j)$ for any i between one and g. The order of x implies

$$0 = [p]_i([p^{n-1}](x)) = pf_i([p^{n-1}(x)) + h_i(([p^{g-1}](x))^p)$$

and so

$$\nu\left(p\,[p^{n-1}]_i(x)\right) \geq \nu\left(([p^{n-1}]_i(x))^p\right).$$

Hence $\nu([p^{g-1}]_i(x)) \leq \frac{\nu(p)}{p-1}$ and so $p^{g-1}\nu(x_j) \leq \frac{\nu(p)}{p-1}$. □

Proposition 4.3. *Let J be the Jacobian of a smooth projective curve X/R of genus g. Let $\Gamma = J(R)_{tors}$. Then the order of the kernel of the reduction map from $J(R)$ to $J_0(k)$ restricted to $J(R)_{tors}$ is bounded by p^{Ng} where N is the smallest integer such that*

$$\frac{\nu(p)}{p^{N+1} - p^N} < 1.$$

Proof. The kernel of the reduction map from $J(R) \to J_0(k)$ is isomorphic to a group $\mathcal{F}(\pi R)$ where $\mathcal{F}$ is a g-dimensional formal group defined over R. If

$x \in \mathcal{F}(\pi R)$ is non-zero element such that $[p^n](x) = 0$ and $[p^{n-1}](x) \neq 0$ namely an element of order p^n, then

$$\nu(x_j) \leq \frac{\nu(p)}{p^n - p^{n-1}}$$

where j is the index between one and g such that $\nu(x_j) = \min\{\nu(x_i) | i = 1, \dots g\}$. However, $\nu(x_j) \geq 1$ and so the maximum order an element of finite order can have is p^N where N is the smallest integer such that $\frac{\nu(p)}{p^{N+1} - p^N} < 1$. Lastly we note that for any abelian variety, the kernel of the multiplication by p^N map is bounded by p^{Ng} [9]. □

Remark 4.4. If $\nu(p) < p - 1$, then the reduction map is injective on $J(R)_{tors}$.

With the propositions in this section and proposition 2.12 it is straightforward to prove the following theorem using the arguments found in [5], [6].

Theorem 4.5. *Let X/R be a smooth projective curve of genus $g \geq 2$ with an R rational point and a non-trivial Kodaira-Spencer class. Then*

$$\#(X(R) \cap \Gamma) \leq p^{(3+N)g}(3^g)[8g-2]g!$$

where $\Gamma = J(R)_{tors}$ and N is the smallest integer such that $\frac{\nu(p)}{p^{N+1} - p^N} < 1$.

Proof. The R rational point of X gives a natural embedding of X into J that extends to a natural embedding of $\alpha : X_0 \to J_0$. We regard X_0 as a closed subvariety of J_0 via this embedding and note that this extends to an inclusion of X_0^1 in J_0^1. The composition of the natural mapping $J_0^1(k) \to J_0(k)$ with $\nabla_0^1 : J(R) \to J_0^1(k)$ is simply the reduction map $J(R) \to J_0(k)$ and consequently by Proposition 4.3 the restriction of ∇_0^1 to Γ has kernel of order at most p^{Ng} where $\frac{\nu(p)}{p^{N+1} - p^N} < 1$.

Let $B = pJ_0^1$. Then the projection $B \to J_0$ is naturally an isogeny whose degree is at most p^{2g}. There is a natural map from $J_0^1(k) \to J_0^1(k)/B(k)$ under which the image of $\nabla_0^1(\Gamma)$ can be at most p^g by Proposition 4.1. Therefore

$$\nabla_0^1(X(R) \cap \Gamma) \subset X_0^1(k) \cap \left[\cup_{i=1}^{p^{2g}} (B + b_i)\right]$$

for $b_1, b_2, \dots, b_{p^{2g}} \in J_0^1$ meaning

$$\#(X(R) \cap \Gamma) \leq p^{Ng} \sum_{i=1}^{p^{2g}} \# \left[X_0^1 \cap (B + b_i)\right].$$

As a translate of an abelian subvariety, $B + b_i$ is complete. By proposition 2.12, X_0^1 is affine. These both are closed subvarieties of J_0^1 and as such their intersection must be finite. It remains to estimate the cardinality of these intersections.

To do this we proceed exactly as in the proof of Theorem 1.11 of [5] adjusting as follows. Recall that X_0^1 and J_0^1 are Zariski locally trivial principal homogenous spaces of the tangent bundles of X_0 and J_0 respectively. Therefore the extensions corresponding to X_0^1 and J_0^1 are

$$0 \to \mathcal{O}_{X_0} \to E_X \to \omega_{X_0} \to 0$$

and

$$0 \to \mathcal{O}_{X_0} \to E_J \to \Omega_{J_0/k} \to 0$$

respectively. The next adjustment is that in our situation the self intersection

$$\left(\mathcal{O}_{\mathbb{P}(E_X)}(1) \cdot \mathcal{O}_{\mathbb{P}(E_X)}(1)\right)_{\mathbb{P}(E_X)} = \deg \omega_{X_0} = 2g-2$$

meaning that

$$\deg_{\mathcal{H}} \mathbb{P}(E_X) = 2g - 2 + 6g = 8g - 2$$

where $\mathcal{H} = \pi_J^* \mathcal{O}_{J_0}(3\Theta) \otimes \mathcal{O}_{\mathbb{P}(E_J)}(1)$ for Θ the theta divisor on J_0. This in turn means

$$\# \left[X_0^1 \cap (B + b_i)\right] \leq (8g-2)(p^g \cdot 3^g \cdot g!)$$

for each $1 \leq i \leq p^{2g}$. The immediate consequence is

$$\#(X(R) \cap \Gamma) \leq p^{Ng} p^{2g} (8g-2)(p^g \cdot 3^g \cdot g!).$$

□

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