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COMPUTING ISOGENY COVARIANT DIFFERENTIAL MODULAR FORMS

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ABSTRACT. We present the computation modulo p^2 and explicit formulas for the unique isogeny covariant differential modular form of order one and weight $\chi_{-p-1,-p}$ called f_{jet}, an isogeny covariant differential modular form of order two and weight $\chi_{-p^2-p,-1,-1}$ denoted by $f_{\text{jet}}h_{\text{jet}}$, and an isogeny covariant differential modular form h_{jet} of order two and weight $\chi_{1-p^2,0,-1}$.

1. INTRODUCTION

In this paper we introduce explicit formulas modulo p^2 for various differential modular forms discussed by Buium in [3], [2], [4]. The central modular form discussed is the unique, up to multiplication by an element in $\mathbb{Z}_p^*$, isogeny covariant differential modular form of order one and weight $\chi_{-p-1,-1}$ called f_{jet}, f_p^1, and f_{jet}^1, respectively, in [3], [2], [4]. For the rest of this paper we mean "unique up to multiplication by an element in $\mathbb{Z}_p^*$" when we say "unique", and we will refer to the unique isogeny covariant differential modular form of order one and weight $\chi_{-p-1,-1}$ by f_{jet}. This modular form has many interesting connections detailed in [1], [3], [2], and [4]. We compute f_{jet} in a p-adic fashion following the construction of f_{jet} detailed in [3] which allows us to compute modulo p^n or specifically modulo p^2. Then we use the explicit formula from this computation to provide modulo p^2 formulas for order two differential modular forms. The specific order two isogeny covariant differential modular forms we describe are $f_{\text{jet}}h_{\text{jet}}$ from [3] also referred to as k_p^2 in [2] or $f_{\text{jet}}^{1,2}$ in [4] and h_{jet} from [3] also referred to as k_p^2/f_p^1 in [2]. We note that modulo p neither of these order two modular forms contain any second order terms.

The strategy is simple. We know that the isogeny covariant differential modular forms $f_{\text{jet}}h_{\text{jet}}$ and h_{jet} of order two and weights $\chi_{-p^2-p,-1,-1}$ and $\chi_{1-p^2,0,-1}$, respectively, are $f_{\text{jet}}h_{\text{jet}} = \phi(f_{\text{jet}})$, where ϕ is the lifting of the Frobenius morphism, and outside the locus, where f_{jet} modulo p is zero $h_{\text{jet}} = \frac{\phi(f_{\text{jet}})}{f_{\text{jet}}}$ [4]. We should note that h_{jet} is defined only outside this zero locus of f_{jet} modulo p. In [5] we have the explicit computation of $\overline{f}_{\text{def}}$ (the p-derivation analog of the Kodaira-Spencer class) which is the reduction modulo p of the unique isogeny covariant differential modular form of weight $\chi_{-p-1,-1}$. By uniqueness, $f_{\text{jet}} \equiv c\overline{f}_{\text{def}}$ modulo p for some $c \in \mathbb{Z}_p^*$. Here we compute f_{jet} directly allowing us to give a formula for the unique

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isogeny covariant differential modular form modulo p^2 and not just modulo p. We also then are able to describe the order two terms that occur in $f_{\text{jet}}h_{\text{jet}}$ and h_{jet} modulo p^2 but not modulo p.

For both context and notation we give the relevant definitions of differential modular forms. Let $p > 3$ be a prime number. Let $M^0 = \mathbb{Z}_p[a_4, a_6, \Delta^{-1}]\hat{}$, $M^1 = \mathbb{Z}_p[a_4, a_6, \delta a_4, \delta a_6, \Delta^{-1}]\hat{}$, and $M^2 = \mathbb{Z}_p[a_4, a_6, \delta a_4, \delta a_6, \delta^2 a_4, \delta^2 a_6, \Delta^{-1}]\hat{}$, where $\Delta = -2^4(4a_4^3 + 27a_6^2)$ and $\mathbb{Z}_p$ is the p-adic integer. We note that a_4, a_6, $\delta a_4, \delta a_6, \delta^2 a_4, \delta^2 a_6$ are variables over $\mathbb{Z}_p$ and that $\hat{}$ represents the p-adic completion. Then the elements of M^1 are called δ modular forms of order one and elements of M^2 are called δ modular forms of order two as defined by Buium in [3].

Recall now that a p-derivation is a set theoretic map, $\delta : A \to B$, from a ring A to an A-algebra B such that

$$\delta(x + y) = \delta x + \delta y + C_p(x, y), \tag{1.1}$$

$$\delta(xy) = y^p \delta x + x^p \delta y + p\delta x \delta y \tag{1.2}$$

for all $x, y \in A$, where $C_p(X, Y) = \frac{X^p + Y^p - (X+Y)^p}{p}$. In Section 2 we will expand these axioms into a more complete list of properties of p-derivations. For now, if A is a complete discrete valuation ring R, where R has maximal ideal generated by p and an algebraically closed residue field k, and if ϕ is the unique lifting of the Frobenius morphism to A, then the p-derivation given by $\delta(x) = (\phi(x) - x^p)/p$ is unique on R.

Now we use the R and δ from our example and set

$$M(R) = \{(a, b) \in R^2 | 4a^3 + 27b^2 \in R^*\}.$$

Then the set $M(R)$ is in one-to-one correspondence with the set of pairs consisting of an elliptic curve over R and an invertible 1-form; namely, each $(\bar{a}_4, \bar{a}_6) \in M(R)$ corresponds to $(E, dx/2y)$, where E is the projective closure of the affine plane curved $y^2 = x^3 + \bar{a}_4 x + \bar{a}_6$. For any $f \in M^1$, if we substitute $a, b, \delta a, \delta b$ in for $a_4, a_6, \delta a_4, \delta a_6$, then f defines a map (still denoted by f) from $M(R)$ to R. This element in M^1 is in fact uniquely determined by the map from $M(R)$ to R. Similar statements are true for $f \in M^2$.

We define a *δ-character* of order ≤ 1 to be a group homomorphism $\chi : R^* \to R^*$ of the form $\chi = \chi_{m,n}$, where

$$\chi_{m,n}(\lambda) = \lambda^m \left(\frac{\phi(\lambda)}{\lambda^p}\right)^n.$$

Then a δ-modular function of order one has weight χ if for any $\lambda \in R^*$

$$f(\lambda^4 a, \lambda^6 b) = \chi(\lambda) f(a, b)$$

for all $(a, b) \in M(R)$. We can easily extend the definition of δ-characters to higher orders. Namely, a δ-character of order ≤ 2 is a group homomorphism $\chi : R^* \to R^*$ of the form $\chi = \chi_{m,n,r}$ where

$$\chi_{m,n,r}(\lambda) = \lambda^m \left(\frac{\phi(\lambda)}{\lambda^p}\right)^n \left(\frac{\phi^2(\lambda)}{\lambda^{p^2}}\right)^r.$$

The criterion for a δ-modular function of order two to have weight χ is exactly the same as the criterion for a δ-modular function of order one to have weight χ. A δ-modular form is a δ-modular function with a weight.

A δ-modular form is *isogeny covariant* if for any two pairs (a,b) and $(\tilde{a},\tilde{b})$ with an etale isogeny of degree N between the corresponding elliptic curves that pulls back $\frac{dx}{y}$ to $\frac{dx}{y}$

$$f(a,b) = N^{-k/2} f(\tilde{a},\tilde{b}),$$

where k is a constant that depends solely on the weight. Note that for $\chi = \chi_{m,n}$ the constant is $k = m + n(1-p)$ and for $\chi = \chi_{m,n,r}$ the constant is $k = m + n(1-p) + r(1-p^2)$.

Theorem 1.1. *The isogeny covariant differential modular form of order one and weight $\chi_{-p-1,-1}$ modulo p^2 is*

$$f_{\text{jet}} = \left[\frac{-72a_6^p\delta a_4 + 48a_4^p\delta a_6}{\Delta^p}\right]\gamma_{2p,p} + h + pH,$$

where $\gamma_{2p,p}$ and h are polynomials in $M_1^0 := M^0 \otimes \mathbb{Z}_p/(p^2)$, H is a polynomial in $M_0^1 := M^1 \otimes \mathbb{Z}_p/(p)$, and H is a nonhomogeneous quadratic in δa_4 and δa_6.

Explicit formulas for h and H are given in Theorem 6.11 and an explicit formula for $\gamma_{2p,p}$ is given in Proposition 6.2.

Theorem 1.2. *The isogeny covariant differential modular form $f_{\text{jet}}h_{\text{jet}}$ of order two and weight $\chi_{-p^2-p,-1,-1}$ modulo p^2 is*

$$f_{\text{jet}}h_{\text{jet}} = \left[\frac{-72a_6^{p^2}(\delta a_4)^p + 48a_4^{p^2}(\delta a_6)^p}{\Delta^{p^2}}\right]\gamma_{2p,p}(a_4^p,a_6^p) + h^* + p\left[\frac{-72a_6^{p^2}\delta^2 a_4 + 48a_4^{p^2}\delta^2 a_6}{\Delta^{p^2}}\right]\gamma_{2p,p}(a_4^p,a_6^p) + pJ,$$

where h^* is a polynomial in $M_1^1 := M^1 \otimes \mathbb{Z}_p/(p^2)$ and J is a polynomial in M_0^1.

Corollary 1.3. *The isogeny covariant differential modular form h_{jet} of order two and weight $\chi_{1-p^2,0,-1}$ modulo p^2 is*

$$h_{\text{jet}} = \frac{\left[-72a_6^{p^2}(\delta a_4)^p + 48a_4^{p^2}(\delta a_6)^p\right]\gamma_{2p,p}(a_4^p,a_6^p) + \Delta^{p^2}h^*}{\Delta^{p^2-p}\left(\left[-72a_6^p\delta a_4 + 48a_4^p\delta a_6\right]\gamma_{2p,p} + \Delta^p h\right)} + p\left(\frac{-\left(\left[-72a_6^{p^2}(\delta a_4)^p + 48a_4^{p^2}(\delta a_6)^p\right]\gamma_{2p,p}(a_4^p,a_6^p) + \Delta^{p^2}h^*\right)H}{\Delta^{p^2-2p}(\left[-72a_6^p\delta a_4 + 48a_4^p\delta a_6\right]\gamma_{2p,p} + \Delta^p h)^2} + \frac{\left[72a_6^{p^2}\delta^2 a_4 + 48a_4^{p^2}\delta^2 a_6\right]\gamma_{2p,p}(a_4^p,a_6^p) + \Delta^{p^2}J}{\Delta^{p^2-p}\left(\left[-72a_6^p\delta a_4 + 48a_4^p\delta a_6\right]\gamma_{2p,p} + \Delta^p h\right)}\right),$$

where h, h^*, H, and J are the same as in Theorems 1.1 and 1.2.

What follows is preliminary information to the calculation of f_{jet}. Let E be the elliptic curve in Weierstrass form over M^0 defined by the homogeneous equation

$$f(X,Y,W) = WY^2 - X^3 - a_4XW^2 - a_6W^3.$$

Let U and V be the affine open subsets of E given by the equations $f(x,y,1)$ and $f(z,1,w)$, respectively. So

$$U = \operatorname{Spec} M^0[X,Y]/(f(X,Y,1)) = \operatorname{Spec} M^0[x,y],$$
$$V = \operatorname{Spec} M^0[Z,W]/(f(Z,1,W)) = \operatorname{Spec} M^0[z,w],$$

and on $U \cap V$

$$z = -x/y,$$
$$w = -1/y,$$

whence $E = U \cup V$. Next we define the first jets of U and V to be the sets

$$U^1 = \operatorname{Spec} M^1[X, Y, \delta X, \delta Y]/(f(X, Y, 1), \delta f(X, Y, 1)) = \operatorname{Spec} M^1[x, y, \delta x, \delta y],$$
$$V^1 = \operatorname{Spec} M^1[Z, W, \delta Z, \delta W]/(f(Z, 1, W), \delta f(Z, 1, W)) = \operatorname{Spec} M^1[z, w, \delta z, \delta w].$$

Then E^1, the first jet space of E, is the gluing of U^1 and V^1 by the maps

$$\begin{aligned} z &= -x/y, \\ w &= -1/y, \\ \delta z &= \frac{x^p \delta y - y^p \delta x}{y^p(y^p + p\delta y)}, \\ \delta w &= \frac{\delta y}{y^p(y^p + p\delta y)}. \end{aligned} \tag{1.3}$$

We can extend the group law on E to a group law on E^1. The group law arises naturally by construction from the group law on E just as E^1 arises naturally by construction from E. This will be detailed explicitly in Section 3.

From now on we will also use the following notation. First by M_n^i we mean $M^i \otimes \mathbb{Z}_p/(p^{n+1})$. For example $M_0^1 = \mathbb{F}_p[a_4, a_6, \delta a_4, \delta a_6, \Delta^{-1}]$, where $\mathbb{F}_p$ is the finite field of p elements. Second by E_m^1 we mean $E^i \otimes M_m^1$, and by $E_m = E_m^0$ we mean $E \otimes M_m^0$. Also we will use $\delta(a_4)$ interchangeably for δa_4, $\delta(a_6)$ interchangeably for δa_6, etc.

To compute f_{jet}, the isogeny covariant δ modular form of weight $\chi_{-p-1,-1}$, we work from [2, Construction 4.1]. The same construction is also described in [4] and [3]. First we find two sections s_U and s_V of the morphisms $U^1 \to U \otimes M^1$ and $V^1 \to V \otimes M^1$, respectively, such that s_U defines a morphism from $U \otimes M^1$ to U^1 and s_V defines a morphism from $V \otimes M^1$ to V^1. Then the difference of the sections under the group law induces a morphism $s_U - s_V : U \cap V \otimes M^1 \to E^1$. Let ζ be the δz coordinate in the difference $s_U - s_V$. By the δz coordinate, we mean the image of $\delta z \in U^1 \cap V^1$ under the ring homomorphism induced by the morphism $s_U - s_V : U \cap V \otimes M^1 \to E^1$. Let $\log_{\mathcal{F}_1^{\phi^1}}(\xi)$ be the formal logarithm of the Frobenius twist of the formal group of the elliptic curve, namely

$$\log_{\mathcal{F}_1^{\phi^1}}(\xi) = \xi + \frac{p\phi(c_1)}{2}\xi^2 + \frac{p^2\phi(c_2)}{3}\xi^3 + \cdots,$$

where the c_i are the coefficients of the power series expansion of the invariant differential [3]. Then $\log_{\mathcal{F}_1^{\phi^1}}(\zeta)$ is a cohomology class in $H^1(E \otimes M^1, \mathcal{O}) \simeq H^1(E, \mathcal{O}) \otimes M^1$, and this resulting class has a representative of the form $\sum a_n y^n + x \sum b_n y^n + x^2 \sum e_n y^n$. The modular form f_{jet} is the coefficient e_{-1} of x^2/y in this representative which is the residue of the cohomology class, namely the image of the cohomology class under the Serre duality pairing.

We will actually work modulo p^2 which means that our end result will be f_{jet} modulo p^2. In fact $f_{\text{jet}} \in M^1$ is a restricted power series whose coefficients expand exponentially in the number of terms in each coefficient of a power of p. Therefore, the formulas necessary to express f_{jet} modulo p^n for $n > 2$ are prohibitive in length. At this point we note that the formal logarithm of the Frobenius twist of the formal group of the elliptic curve modulo p^2 is in fact the identity. This certainly simplifies one step of the computation modulo p^2; however, for $n > 4$ this formal logarithm is no longer trivial modulo p^n, meaning this step is not trivial for large n. As a preliminary step to computing f_{jet} we detail some computation guidelines for p-derivations and the group law for E^1 modulo p^2.

2. Properties of p-derivations

Recall that a p-derivation is a set theoretic map, $\delta : A \to B$, from a ring A to an A-algebra B with $\delta(1) = 0$ such that

$$\delta(x+y) = \delta x + \delta y + C_p(x,y), \qquad \delta(xy) = y^p\delta x + x^p\delta y + p\delta x\delta y$$

for all $x, y \in A$, where $C_p(X,Y) = \frac{X^p+Y^p-(X+Y)^p}{p}$. In the case when $A = B = R$, where R is a complete discrete valuation ring with maximal ideal generated by p and has an algebraically closed residue field, there is a unique p-derivation given by $\delta(x) = (\phi(x) - x^p)/p$, where ϕ is the unique lifting of the Frobenius morphism to R.

This definition implies that if $\varphi : A \to B$ is the ring homomorphism associated to B being an A-algebra, then

$$(\varphi, \delta) : A \to W_2(B)$$

is a ring homomorphism, where $W_2(B)$ is the ring of *Witt vectors* of length two on B. With (φ, δ) as above, $\phi : A \to B$ defined by $\phi(x) = \varphi(x)^p + p\delta(x)$ is a ring homomorphism. In case $B = A$, this is a lifting of the Frobenius endomorphism $F(x) = x^p$ of A/pA.

While no further axioms for p-derivations beyond those in the definition are necessary for computation, the following p-derivation rules are very convenient for computation. Before introducing these rules we must define an extension of $C_p(X,Y)$.

Definition 2.1. For any $\sum q$, let

$$C_p^{\text{ext}}(\sum q) = \frac{\sum q^p - (\sum q)^p}{p}.$$

Note that $C_p^{\text{ext}}(X+Y) = \frac{X^p+Y^p-(X+Y)^p}{p} = C_p(X,Y)$; thus, this is a very natural definition.

Lemma 2.2. *Let $\delta : A \to B$ be a p-derivation, let $g = \sum q, x, y \in A$, and let $n > 0$ be an integer. Then the following are true.*

1. $\delta(\sum q) = \sum \delta q + C_p^{\text{ext}}(\sum q)$.
2. $\delta(-1) = 0$.
3. $\delta(-x) = -\delta x$.
4. $\delta(x^n) = \sum_{k=1}^{n} \binom{n}{k} p^{k-1} x^{(n-k)p}(\delta x)^k = \dfrac{-x^{np} + (x^p + p\delta x)^n}{p}$.
5. $\delta\left(\dfrac{1}{x}\right) = \dfrac{-\delta x}{x^p(x^p + p\delta x)}$.
6. $\delta\left(\dfrac{y}{x}\right) = \dfrac{x^p\delta y - y^p\delta x}{x^p(x^p + p\delta x)}$.

3. The group law for the first p-jet space of E

We now want to make the group law on E^1 explicit. This is necessary since the main result requires us to subtract two sections using the group law. The group law on the first p-jet is induced by the group law on the elliptic curve E, so we start by giving the group law on E. Let ρ and ψ be the equations that define the group law on E. So if $(z_1, w_1) \oplus (z_2, w_2) = (z_3, w_3)$, then

$$\begin{aligned} z_3 &= \rho(z_1, w_1, z_2, w_2), \\ w_3 &= \psi(z_1, w_1, z_2, w_2). \end{aligned}$$

Then the group law on E^1 is an extension of the group law on E such that if $(z_1, w_1, \delta z_1, \delta w_1) \oplus (z_2, w_2, \delta z_2, \delta w_2) = (z_3, w_3, \delta z_3, \delta w_3)$, then

$$\begin{aligned} z_3 &= \rho(z_1, w_1, z_2, w_2), \\ w_3 &= \psi(z_1, w_1, z_2, w_2), \\ \delta z_3 &= \delta(\rho(z_1, w_1, z_2, w_2)), \\ \delta w_3 &= \delta(\psi(z_1, w_1, z_2, w_2)). \end{aligned}$$

To find appropriate ρ and ψ, we must consider actual formulas for the group law. On the elliptic curve E, the group law can be explicitly formulated using the chord-tangent approach. In this approach we consider that every line intersects the elliptic curve at exactly three points counting multiplicity. We choose a specific point, O, to be the origin; in this case the point we choose to be the origin is the point at infinity, $(0, 1, 0)$. We then define the inverse of a point P to be the third point on the line that intersects P and the origin. We denote this point by $-P$. So if we want to add $P \oplus Q$, we take the line through P and Q and let R be the third point on the line. Then we define $P \oplus Q = -R$. This definition arises naturally from the theory of Weil divisors. We refer to the case when $P = Q$ as the tangent case and $P \neq Q$ as the tangent case. From now on we will focus on the chord case of the chord-tangent approach since that is the most general case and the case used when computing the group law for a p-jet space.

We use the standard procedure for finding explicit formulas for group law in the z and w coordinates. In these coordinates our origin is $(0, 0)$. To start with, we recall data on V; namely, that $f(z, 1, w) = w - z^3 - a_4 z w^2 - a_6 w^3$ is the curve we will be using and that any line through $P = (z_0, w_0)$ and the origin of $(0, 0)$ will intersect $f(z, 1, w)$ at the third point $(-z_0, -w_0)$. Whence $-P = (-z_0, -w_0)$.

Now consider two points P_1 and P_2 denoted by (z_i, w_i) for $i = 1, 2$, respectively. If we assume that $z_1 \neq z_2$, the line connecting these two points is

$$w = \frac{w_2 - w_1}{z_2 - z_1}(z - z_1) + w_1.$$

To find the sum $P_1 \oplus P_2$, we must find the three points counting multiplicity of the intersection of this line with the curve $f(z, 1, w)$. If we substitute the line into the curve $f(z, 1, w)$ we get a cubic equation in terms of z. Finding these three points becomes a matter of finding the roots of the resulting cubic equation. On the other hand, we already know two of the roots, namely $z = z_1$ and $z = z_2$. The third root is

$$z = \frac{-2w_2z_1 - w_1z_1 + 2w_1z_2 - 3a_6w_2w_1^2z_2 - a_4w_1^2z_2^2 + w_2z_2 + 3w_2^2z_1a_6w_1 + w_2^2z_1^2a_4}{3a_6w_2w_1(w_2-w_1) + 3z_2z_1(z_2-z_1) + a_4(w_2^2z_1 - w_1^2z_2) + w_1 - w_2 + 2a_4w_2w_1(z_2-z_1)}.$$

So for $z_1 \neq z_2$, the $P_3 = P_1 \oplus P_2$ has coordinates

$$z_3 = -\frac{-2w_2z_1 - w_1z_1 + 2w_1z_2 - 3a_6w_2w_1^2z_2 - a_4w_1^2z_2^2 + w_2z_2 + 3w_2^2z_1a_6w_1 + w_2^2z_1^2a_4}{3a_6w_2w_1(w_2-w_1) + 3z_2z_1(z_2-z_1) + a_4(w_2^2z_1 - w_1^2z_2) + w_1 - w_2 + 2a_4w_2w_1(z_2-z_1)},$$

$$w_3 = -\frac{3w_2z_2z_1^2 + z_1w_2^2w_1a_4 - 3z_1w_1z_2^2 + w_1^2 - w_2w_1^2z_2a_4 - w_2^2}{3a_6w_2w_1(w_2-w_1) + 3z_2z_1(z_2-z_1) + a_4(w_2^2z_1 - w_1^2z_2) + w_1 - w_2 + 2a_4w_2w_1(z_2-z_1)}.$$

From this information, if we want the formulation of group law on E^1 we must simply take the p-derivation of these equations. The resulting group law for $P_i = (z_i, w_i, \delta z_i, \delta w_i) \in V^1$ is

$$z_3 = -\frac{\alpha}{\mu},$$

$$w_3 = -\frac{\beta}{\mu},$$

$$\delta z_3 = -\frac{\mu^p \delta\alpha - \alpha^p \delta\mu}{\mu^p(\mu^p + p\delta\mu)},$$

$$\delta w_3 = -\frac{\mu^p \delta\beta - \beta^p \delta\mu}{\mu^p(\mu^p + p\delta\mu)},$$

where $P_3 = P_1 \oplus P_2$,

$$\begin{aligned}\alpha &= -2w_2z_1 - w_1z_1 + 2w_1z_2 - 3a_6w_2w_1^2z_2 \\ &\quad - a_4w_1^2z_2^2 + w_2z_2 + 3w_2^2z_1a_6w_1 + w_2^2z_1^2a_4,\end{aligned}$$

$$\beta = 3w_2z_2z_1^2 + z_1w_2^2w_1a_4 - 3z_1w_1z_2^2 + w_1^2 - w_2w_1^2z_2a_4 - w_2^2,$$

$$\begin{aligned}\mu &= 3a_6w_2w_1(w_2 - w_1) + 3z_2z_1(z_2 - z_1) \\ &\quad + a_4(w_2^2z_1 - w_1^2z_2) + w_1 - w_2 + 2a_4w_2w_1(z_2 - z_1),\end{aligned}$$

and $\delta\alpha$, $\delta\beta$, $\delta\mu$ are the respective p-derivatives which are not included here because of their lengthy nature. On the other hand, this group law also describes the group law on E^1_m. For example, if $m = 1$, then we consider this same group law modulo p^2. This does shorten the expressions of $\delta\alpha$, $\delta\beta$, and $\delta\mu$ for $m \leq 5$ to lengths that are possible to work with in computer algebra systems.

Besides shortening the expressions for α, β, and μ one other advantage of explicitly detailing the group law on E^1_1 rather than E^1 is that we may write δz_3 and δw_3 in terms of polynomials in $\delta\alpha$, $\delta\beta$, and $\delta\mu$ by using their series expansions. Hence we have the following description for the group law on E_1^1:

$$
\begin{aligned}
z_3 &= -\frac{\alpha}{\mu},\\
w_3 &= -\frac{\beta}{\mu},\\
\delta z_3 &= \frac{1}{\mu^{3p}}(-\mu^p\delta\alpha + \alpha^p\delta\mu)(\mu^p - p\delta\mu),\\
\delta w_3 &= \frac{1}{\mu^{3p}}(-\mu^p\delta\beta + \beta^p\delta\mu)(\mu^p - p\delta\mu),
\end{aligned}
$$

where $\delta\alpha$, $\delta\beta$, and $\delta\mu$ are now expressions modulo p^2.

4. The section on U that defines a map from $U \otimes M^1$ to U^1 and the section on V that defines a map from $V \otimes M^1$ to V^1

We in fact want a specific map from U to U^1; namely, the morphism which takes δx and δy to elements such that $\delta f(x, y, 1)$ is mapped to 0. To do this we use a variant of Hensel's Lemma involving two variables which will be illuminated as we go along. To find the appropriate δx and δy, we first consider the explicit expression of $\delta f(x, y, 1)$,

(4.1)
$$
\begin{aligned}
&-p^2\delta x^3 + (-3x^p\delta x^2 - \delta a_4\delta x + \delta y^2)p\\
&\qquad + (-3x^{2p} - a_4^p)\delta x + 2y^p\delta y - \delta a_4 x^p - \delta a_6 + C_p^{\mathrm{ext}}(y^2 - x^3 - a_4x - a_6),
\end{aligned}
$$

and from this polynomial define $P_{U,0} = -\delta a_4 x^p - \delta a_6 + C_p^{\mathrm{ext}}(y^2 - x^3 - a_4x - a_6)$. From now on for convenience of notation we will denote $f(x, y, 1)$ simply by f. We will also denote $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ by f_x and f_y, respectively.

Now we let A and B be polynomials in $M^0[x, y]$ such that $Af_x + Bf_y = 1$. Specifically

$$
\begin{aligned}
A &= \frac{2^4(4a_4^2 + 6x^2a_4 - 9xa_6)}{\Delta},\\
B &= \frac{2^3(9y)(2xa_4 - 3a_6)}{\Delta}.
\end{aligned}
$$

By simple arithmetic, $A^pf_x^p + B^pf_y^p = 1 + A^pf_x^p + B^pf_y^p - (Af_x + Bf_y)^p = 1 + pC_p^{\mathrm{ext}}(Af_x + Bf_y)$. Now we consider the relationship between f_x^p, f_y^p and the coefficients of δx and δy, respectively. First recall that $n = n^p + p\delta(n)$ for any positive integer n. So we can write the coefficients of δx and δy from equation (4.1) as

$$
\begin{aligned}
\text{Coefficient of } \delta x &= -3x^{2p} - a_4^p = -(3^p + p\delta(3))x^{2p} - a_4^p\\
&= -p\delta(3)x^{2p} + f_x^p + pC_p^{\mathrm{ext}}(-3x^2 - a_4)\\
&= f_x^p + p(-\delta(3)x^{2p} + C_p^{\mathrm{ext}}(-3x^2 - a_4)),\\
\text{Coefficient of } \delta y &= 2y^p = (2^p + p\delta(2))y^p\\
&= f_y^p + p\delta(2)y^p.
\end{aligned}
$$

Then combining these with the equation $A^p f_x^p + B^p f_y^p = 1 + pC_p^{\text{ext}}(Af_x + Bf_y)$,

$$\begin{aligned}A^p&(-3x^{2p} - a_4^p) + B^p(2y^p)\\ &= A^p(f_x^p + p(-\delta(3)x^{2p} + C_p^{\text{ext}}(-3x^2 - a_4))) + B^p(f_y^p + p\delta(2)y^p)\\ &= 1 + p\left(C_p^{\text{ext}}(Af_x + Bf_y) + A^p(-\delta(3)x^{2p} + C_p^{\text{ext}}(-3x^2 - a_4)) + B^p\delta(2)y^p\right).\end{aligned}$$

Now if we let $R_{U,0} = C_p^{\text{ext}}(Af_x+Bf_y)+A^p(-\delta(3)x^{2p}+C_p^{\text{ext}}(-3x^2-a_4))+B^p\delta(2)y^p$, then $A^p(-3x^{2p} - a_4^p) + B^p(2y^p) = 1 + pR_{U,0}$.

With the computations in the previous paragraph we now have enough tools to perform the iteration step in Hensel's Lemma. We assume that

$$\begin{aligned}\delta x &= -P_{U,0}A^p + p\eta,\\ \delta y &= -P_{U,0}B^p + p\sigma,\end{aligned}$$

and plug these assumptions into equation (4.1). Then we solve the resulting equation for η and σ, keeping in mind that we are working modulo p^2. (Note: The procedure is the same working modulo p^3 etc., but in that case one must assume a p^2 term for the δx and δy and then perform the iteration twice.)

$$\begin{aligned}(-3x^p\delta x^2 - \delta a_4\delta x + \delta y^2)p &+ (-3x^{2p} - a_4^p)\delta x + 2y^p\delta y + P_{U,0}\\ = p(&-3x^pP_{U,0}^2A^{2p} + \delta a_4P_{U,0}A^p + P_{U,0}^2B^{2p}\\ &- P_{U,0}R_{U,0} + (-3x^{2p} - a_4^p)\eta + 2y^p\sigma).\end{aligned}$$

Now if we let $P_{U,1} = -3x^pP_{U,0}^2A^{2p} + \delta a_4P_{U,0}A^p + P_{U,0}^2B^{2p} - P_{U,0}R_{U,0}$, then

$$\begin{aligned}\eta &= -P_{U,1}A^p,\\ \sigma &= -P_{U,1}B^p.\end{aligned}$$

So the morphism that takes x to x, y to y, δx to $-P_{U,0}A^p - pP_{U,1}A^p$, and δy to $-P_{U,0}B^p - pP_{U,1}B^p$ will map δf to 0. Our corresponding section, s_U, is

$$(x, y, -A^p(P_{U,0} + pP_{U,1}), -B^p(P_{U,0} + pP_{U,1})).$$

Next we find the section s_V that defines a specific map from V to V^1 such that under this map $\delta f(z, 1, w)$ is taken to 0. Since the techniques used are identical to those used to find s_U, we will omit most of the details. From now on for convenience of notation we will refer to $f(z, 1, w)$ as g, and $\frac{\partial g}{\partial z}$, $\frac{\partial g}{\partial w}$ will be referred to as g_z and g_w, respectively.

Let $P_{V,0} = -\delta a_6w^{3p} - \delta a_4z^pw^{2p} + C_p^{\text{ext}}(w - z^3 - a_4zw^2 - a_6w^3)$ and let C and D be polynomials in $M^0[z, w]$ such that $Cg_z + Dg_w = 1$. Specifically

$$C = z(-\frac{3}{2}a_6w - a_4z), \qquad D = -\frac{3}{2}a_6w^2 - wa_4z + 1.$$

Next let

$$R_{V,0} = C_p^{\text{ext}}(Cg_z + Dg_w) + C^p(-\delta(3)z^{2p} + C_p^{\text{ext}}(-3z^2 - a_4w^2)) + D^p(-\delta(3)a_6^p w^{2p} - \delta(2)a_4^p z^p w^p + C_p^{\text{ext}}(1 - 3a_6w^2 - 2a_4zw))$$

and let

$$P_{V,1} = -3z^p(P_{V,0}C^p)^2 + (2a_4^p w^p(-P_{V,0}D^p) + \delta a_4 w^{2p})(P_{V,0}C^p) - (a_4^p z^p + 3a_6^p w^p)(P_{V,0}D^p)^2 - (3\delta a_6 w^{2p} + 2\delta a_4 z^p w^p)(-P_{V,0}D^p) - P_{V,0}R_{V,0}.$$

Then the section s_V defining a map from V to V^1 is

$$(z, w, -C^p(P_{V,0} + pP_{V,1}), -D^p(P_{V,0} + pP_{V,1})).$$

5. $s_U - s_V$ under the group law

We now need the δz coordinate also referred to as ζ in the difference, $s_U - s_V$, of these two sections under the group law. We will work with the element (z, w, z', w') where $z' = \delta z$ and $w' = \delta w$. Recall from the Introduction that our gluing maps on the intersection $U^1 \cap V^1$ are

$$z = -x/y,$$
$$w = -1/y,$$
$$\delta z = \frac{x^p \delta y - y^p \delta x}{y^p(y^p + p\delta y)},$$
$$\delta w = \frac{\delta y}{y^p(y^p + p\delta y)}.$$

So if we let $x' = \delta x$ and $y' = \delta y$, then in terms of the coordinates on U^1, our element is $(-x/y, -1/y, \frac{x^p y' - y^p x'}{y^p(y^p+py')}, \frac{y'}{y^p(y^p+py')})$, which modulo p^2 is the same as $(-x/y, -1/y, \frac{1}{y^{3p}}(x^p y' - y^p x')(y^p - py'), \frac{1}{y^{3p}}(y')(y^p - py'))$. Then under the map s_U, this element is mapped to

$$\left(-x/y, -1/y, \frac{-x^p B^p(P_{U,0} + pP_{U,1}) + y^p A^p(P_{U,0} + pP_{U,1})}{y^p(y^p - pB^p(P_{U,0} + pP_{U,1}))}, \frac{-B^p(P_{U,0} + pP_{U,1})}{y^p(y^p - pB^p(P_{U,0} + pP_{U,1}))}\right)$$

which simplifies modulo p^2 to

$$\left(-x/y, -1/y, \frac{(-x^p B^p + y^p A^p)(y^p P_{U,0} + p(B^p P_{U,0}^2 + y^p P_{U,1}))}{y^{3p}}, \frac{-B^p(y^p P_{U,0} + p(B^p P_{U,0}^2 + y^p P_{U,1}))}{y^{3p}}\right).$$

Under the map s_V the element (z, w, z', w') is mapped to

$$(z, w, -C^p(P_{V,0} + pP_{V,1}), -D^p(P_{V,0} + pP_{V,1})).$$

The image of the element (z, w, z', w') under the difference map $s_U - s_V$ is the difference under the group law on E^1 of the image of (z, w, z', w') under s_U and the image of (z, w, z', w') under s_V. In order to take the difference we must first take the inverse under the group law of the image of (z, w, z', w') under s_V, which is

$$(-z, -w, C^p(P_{V,0} + pP_{V,1}), D^p(P_{V,0} + pP_{V,1}))$$

and then add this to the image of (z, w, z', w') under s_U. Specifically we will let

$$\begin{aligned}
z_1 &= -x/y = z,\\
w_1 &= -1/y = w,\\
\delta z_1 &= \frac{(-x^p B^p + y^p A^p)(y^p P_{U,0} + p(B^p P_{U,0}^2 + y^p P_{U,1}))}{y^{3p}}\\
&= (w^p z^p B^p + w^p A^p)(-P_{U,0} + p(w^p B^p P_{U,0}^2 - P_{U,1})),\\
\delta w_1 &= \frac{-B^p(y^p P_{U,0} + p(B^p P_{U,0}^2 + y^p P_{U,1}))}{y^{3p}}\\
&= w^{2p} B^p(-P_{U,0} + p(w^p B^p P_{U,0}^2 - P_{U,1})),\\
z_2 &= -z,\\
w_2 &= -w,\\
\delta z_2 &= C^p(P_{V,0} + pP_{V,1}),\\
\delta w_2 &= D^p(P_{V,0} + pP_{V,1}),
\end{aligned}$$

and apply the explicit formulation of the group law detailed in Section 3. Also since for the purpose of our computation we only need the δz_3 term, this is the only one we will formulate in detail.

We are going to be analyzing $\frac{1}{\mu^{3p}}(-\mu^p\delta\alpha + \alpha^p\delta\mu)(\mu^p - p\delta\mu)$ with the above terms substituted in for z_1, w_1, etc. When we do this,

$$\begin{aligned}
\alpha &= 0,\\
\mu &= 2w + 6a_6 w^3 + 6a_4 z w^2 + 6z^3,
\end{aligned}$$

which, if we then add $6f(z, 1, w)$, we have $\mu = 8w = -8/y$.

Proposition 5.1. *The $\zeta = \delta z$ coordinate of $s_U - s_V$ is*

$$\zeta = \frac{-\delta\alpha}{(8w)^p} + p\frac{\delta\alpha\delta\mu}{(8w)^{2p}},$$

where the above expressions are used for z_1, z_2, δz_1, δz_2, δw_1, etc. in $\delta\alpha$ and $\delta\mu$.

The next step in the computation of f_{def} is to apply the formal logarithm of the Frobenius twist of the formal group of the elliptic curve to ζ. This is a triviality as mentioned in the Introduction by the following proposition.

Proposition 5.2. *Let $\log_{\mathcal{F}_1^{\phi^1}}(\xi)$ be the formal logarithm of the Frobenius twist of the formal group of the elliptic curve. Then*

$$\log_{\mathcal{F}_1^{\phi^1}}(\zeta) = \zeta \text{ modulo } p^2.$$

Proof. Recall

$$\log_{\mathcal{F}_1^{\phi^1}}(\xi) := \xi + \frac{p\phi(c_1)}{2}\xi^2 + \frac{p^2\phi(c_2)}{3}\xi^3 + \cdots,$$

where the c_i are the coefficients of the power series expansion of the invariant differential [3, p. 127]. From [6, p. 113] we know that the invariant differential

$$\omega(z) = (1 + 2a_4 z^4 + \cdots)dz$$

and so $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 2a_4 \cdots$. However, the power of p in p^n/n is at least 2 for all $n \geq 4$. Hence, modulo p^2 the power series $\log_{\mathcal{F}_1^{\phi^1}}(\xi)$ is the identity. □

6. Residue of the cohomology class

Recall that any cohomology class in $H^1(E \otimes M^1, \mathcal{O}) \simeq H^1(E, \mathcal{O}) \otimes M^1$ has a representative of the form $\sum a_n y^n + x \sum b_n y^n + x^2 \sum e_n y^n$. Let us refer to the coefficient e_{-1} in a sum $\sum a_n y^n + x \sum b_n y^n + x^2 \sum e_n y^n$ as the residue of the sum. The final step in the computation of f_{jet} modulo p^2 is to take the residue of $\zeta = \delta z$ coordinate of $s_U - s_V$, which is a cohomogy class as a result of the above proposition.

While the idea behind taking the residue is simple, namely write ζ as $\sum a_n y^n + x \sum b_n y^n + x^2 \sum e_n y^n$ and take the coefficient of x^2/y in this sum which is e_{-1}, the practice is computationally unfeasible. Instead we break the process of finding the residue of ζ into parts. The residue map has some useful properties; namely, it is linear and the residue of any function that is regular on U or any function that is regular on V is zero. So we can take the residue of the terms in ζ and then add together the result to get the residue of ζ.

As a preliminary step to the task of analyzing the residue of the terms of $\zeta = \delta z$, we write the following expressions in both coordinates of U and coordinates of V:

$$
\begin{aligned}
A &= \frac{2^4(4a_4^2 + 6x^2 a_4 - 9xa_6)}{\Delta} = \frac{2^4(4a_4^2 w^2 + 6z^2 a_4 - 9zwa_6)}{w^2 \Delta}, \\
B &= \frac{2^3(9y)(2xa_4 - 3a_6)}{\Delta} = -\frac{2^3(9)(2za_4 - 3wa_6)}{w^2 \Delta}, \\
C &= z(-\frac{3}{2}a_6 w - a_4 z) = \frac{x}{y^2}\left(-\frac{3}{2}a_6 - a_4 x\right), \\
D &= -\frac{3}{2}a_6 w^2 - wa_4 z + 1 = \frac{1}{y^2}\left(x^3 - \frac{1}{2}a_6\right), \\
P_{U,0} &= \frac{P_{V,0}}{w^{3p}}.
\end{aligned}
$$

We also note that $P_{U,0}$ and $P_{U,1}$ are regular on U, and that $P_{V,0}$, $P_{V,1}$, and $C_p^{\text{ext}}(3a_6w^3 + 3a_6w^3 + 3z^3 + 3z^3 + a_4w^2z + a_4w^2z + w + w + 2a_4w^2z + 2a_4w^2z)$ are regular on V. So as an example the following combinations of taken from $\frac{-\delta\alpha}{(8w)^p}$ are regular on V:

$$
\frac{-p3^p D^p C^p a_6^p w^p P_{V,0}^2}{8^p}, \qquad \frac{p2w^p a_4^p z^p C^p P_{V,1}}{8^p}, \qquad \frac{-C^p P_{V,0}}{8^p}.
$$

More examples of regular combinations on V, this time taken from $\frac{\delta\alpha\delta\mu}{(8w)^{2p}}$, are

$$
\frac{(3^p w^{3p} C^p a_6^p P_{V,0})(3(3^p)a_6^p w^{4p} B^p P_{U,0})}{(8w)^{2p}} \quad \text{and} \quad \frac{(-3^p w^{4p} a_6^p A^p P_{U,0})(2(3^p)\delta(a_6)w^{3p})}{(8w)^{2p}}.
$$

Since the residue of terms that are regular on either U or V is zero, we can exclude these terms from consideration in computing the residue class of ζ. This leads to the following proposition in which for brevity's sake we let

$$
\Upsilon = C_p^{\text{ext}}(y^2 - x^3 - a_4x - a_6).
$$

Proposition 6.1. *The residue of ζ is equal to the residue of*

$$\left(\frac{(1-2^p)A^p}{y^p}+\frac{-2x^pB^p}{y^{2p}}+\frac{(-3^pa_6^p-2a_4^px^p)A^p}{y^{3p}}+\frac{(-1-2^p)x^pD^p}{y^{3p}}\right)\frac{P_{U,0}}{8^p}$$
$$+p\Bigg(F_1C_p^{\text{ext}}(3a_6w^3+3a_6w^3+3z^3+3z^3+a_4w^2z$$
$$+a_4w^2z+w+w+2a_4w^2z+2a_4w^2z)$$
$$+F_2C_p^{\text{ext}}(Af_x+Bf_y)+F_3C_p^{\text{ext}}(-3x^2-a_4)$$
$$+F_4\Upsilon^2+(F_5+F_6+F_7)\Upsilon+F_8\Bigg),$$

where F_i are polynomials in $M_1^1[x^p,y^p,\Upsilon]$.

Proof. This is proved by the very precise removal of almost all regular terms using a computer algebra system. □

It is now necessary to compute residues of terms whose residue may be nontrivial. Namely, we provide a formula for the residue of $\frac{x^a}{y^b}$ which we will call $\gamma_{a,b}$. We let $\binom{n}{k}$ denote the binomial coefficient with the convention that $\binom{n}{k}=0$ if $k>n$. Then from [5] we know

Proposition 6.2. *Let a and b be positive integers. Let m and $n\in\{0,1,2\}$ be integers such that $a=3m+n$. Then the residue of $\frac{x^a}{y^b}$ is*

$$\gamma_{a,b}=\begin{cases}0 & \text{if } b \text{ is even,}\\ \displaystyle\sum_{k=0}^{\infty}\binom{m+k}{3k+2-n}\binom{m-2k-2+n}{\frac{b-1}{2}}(-1)^{m+k-\frac{b-1}{2}}(a_4)^{3k+2-n}(a_6)^{m-2k-2+n-\frac{b-1}{2}} & \text{if } b \text{ is odd.}\end{cases}$$

Obviously, because of the convention for binomial coefficients, there will be integers a and b with b odd for which $\gamma_{a,b}$ is 0. In fact, if $\frac{3b}{2}>a$, $\gamma_{a,b}=0$ because of the binomial coefficient $\binom{m-2k-2+n}{\frac{b-1}{2}}$. We now introduce a series of propositions that are just expanded formulas for expressions found in Proposition 6.1.

Proposition 6.3.

$$\Upsilon=C_p^{\text{ext}}(y^2-x^3-a_4x-a_6)$$
$$=\frac{1}{p}\left[\left(\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k\right)y^{2p}-\sum_{k=1}^{p-1}\binom{p}{k}x^{3k}(a_4x+a_6)^{p-k}-\sum_{k=1}^{p-1}\binom{p}{k}a_4^ka_6^{p-k}x^k\right].$$

Proposition 6.4.

$$A^p=\frac{2^{4p}(4a_4^2+6x^2a_4-9xa_6)^p}{\Delta^p}$$
$$=\frac{2^{4p}}{\Delta^p}\Bigg[4^pa_4^{2p}+6^px^{2p}a_4^p-9^px^pa_6^p$$
$$+\sum_{k=1}^{p-1}\binom{p}{k}(4a_4^2)^{p-k}(6x^2a_4-9xa_6)^k+\sum_{k=1}^{p-1}\binom{p}{k}(6x^2a_4)^k(-9xa_6)^{p-k}\Bigg].$$

Proposition 6.5.

$$B^p = \frac{2^{3p}(9y)^p(2xa_4 - 3a_6)^p}{\Delta^p}$$
$$= \frac{2^{3p}(9y)^p}{\Delta^p}\left[2^p x^p a_4^p - 3^p a_6^p + \sum_{k=1}^{p-1}\binom{p}{k}(2xa_4)^k(-3a_6)^{p-k}\right].$$

Proposition 6.6.

$$C_p^{\text{ext}}(3a_6w^3 + 3a_6w^3 + 3z^3 + 3z^3 + a_4w^2z + a_4w^2z + w + w + 2a_4w^2z + 2a_4w^2z)$$
$$= \frac{1}{p}\left[2(3a_6)^p + 2(3x^3)^p + 2(1+2^p)(a_4x)^p + (2-8^p)y^{2p}\right]\left(\frac{-1}{y^{3p}}\right).$$

Proposition 6.7.

$$C_p^{\text{ext}}(Af_x + Bf_y)$$
$$= \frac{1}{p}\sum_{k=1}^{p-1}\sum_{i=0}^{k}\sum_{j=0}^{p-i}\binom{p}{k}\binom{k}{i}\binom{p-i}{j}(-1)^{k-i}$$
$$\times \left(\frac{9(2^4)}{\Delta}\right)^{p-i}(2a_4)^j(-3a_6)^{p-i-j}x^jy^{2(p-i)}.$$

Proposition 6.8.

$$C_p^{\text{ext}}(-3x^2 - a_4) = -\frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k}3^k a_4^{p-k}x^{2k}.$$

Using this series of propositions, we can now explicitly write down the residue of ζ by computing the residue of the formula in Proposition 6.1. First we introduce some more notation.

Definition 6.9. Define $\mu_{a,b}$ to be the residue of $\frac{x^a\Upsilon}{y^b}$ where

$$\Upsilon = C_p^{\text{ext}}(y^2 - x^3 - a_4x - a_6).$$

Definition 6.10. Define $\tau_{a,b}$ to be the residue of $\frac{x^a\Upsilon^2}{y^b}$ where

$$\Upsilon = C_p^{\text{ext}}(y^2 - x^3 - a_4x - a_6).$$

Using the formulas above, it is easy to check that for some, but not all, values, $\mu_{a,b}$ will be zero. Similarly there are values of a and b for which $\tau_{a,b}$ is nonzero and for which it is zero. Some examples of $\mu_{a,b}$ are

$$\mu_{p,3p} = 0,$$
$$\mu_{p,p} = \frac{1}{p}\left[-\sum_{k=1}^{p-1}\binom{p}{k}\sum_{i=0}^{p-k}\binom{p-k}{i}a_4^ia_6^{p-k-i}\gamma_{p+3k+i,p} - \sum_{k=1}^{p-1}\binom{p}{k}a_4^ka_6^{p-k}\gamma_{p+k,p}\right].$$

Note that both $\gamma_{a,b}$ and $\mu_{a,b}$ are in M_1^0. We can now prove the following theorem.

Theorem 6.11. *The reduction modulo p^2 of $f_{\rm jet}$ is*

$$\left[\frac{9^p(2^p-4^p-2(3^p))a_6^p\delta(a_4)}{\Delta^p}+\frac{2^p(-6^p+12^p+2(9^p))a_4^p\delta(a_6)}{\Delta^p}\right]\gamma_{2p,p}$$
$$+\frac{1}{\Delta^p}\Big[2^p(1-2^p)4^pa_4^{2p}\mu_{0,p}+(-18^p(1-2^p)+2(27^p))a_6^p\mu_{p,p}$$
$$+(12^p(1-2^p)-2(18^p))a_4^p\mu_{2p,p}+(2(18^p)-36^p)a_4^pa_6^p\mu_{2p,3p}-2(12^p)a_4^{2p}\mu_{3p,3p}\Big]$$
$$+p\Big(H_0+H_1+H_2+H_3+H_4+H_5+H_6+H_7+H_8\Big)$$

where H_0 is

$$\frac{1}{p}\Bigg((1-2^p)\frac{2^p}{\Delta^p}\Big[-\delta(a_4)6^pa_4^p\gamma_{3p,p}+\sum_{k=1}^{p-1}\binom{p}{k}(4a_4^2)^{p-k}\sum_{i=0}^{k}\binom{k}{i}(6a_4)^i(-9a_6)^{k-i}$$
$$\times\big(-\delta(a_4)\gamma_{p+k+i,p}-\delta(a_6)\gamma_{k+i,p}+\mu_{k+i,p}\big)$$
$$+\sum_{k=1}^{p-1}\binom{p}{k}(6a_4)^k(-9a_6)^{p-k}\big(-\delta(a_4)\gamma_{2p+k,p}-\delta(a_6)\gamma_{p+k,p}+\mu_{p+k,p}\big)\Big]$$
$$-2\frac{9^p}{\Delta^p}\Big[-\delta(a_4)2^pa_4^p\gamma_{3p,p}$$
$$+\sum_{k=1}^{p-1}\binom{p}{k}(2a_4)^k(-3a_6)^{p-k}\big(-\delta(a_4)\gamma_{2p+k,p}-\delta(a_6)\gamma_{p+k,p}+\mu_{p+k,p}\big)\Big]$$
$$-(3^pa_6^p)\frac{2^p}{\Delta^p}\Big[\sum_{k=1}^{p-1}\binom{p}{k}(4a_4^2)^{p-k}\sum_{i=0}^{k}\binom{k}{i}(6a_4)^i(-9a_6)^{k-i}\mu_{k+i,3p}$$
$$+\sum_{k=1}^{p-1}\binom{p}{k}(6a_4)^k(-9a_6)^{p-k}\mu_{p+k,3p}\Big]$$
$$-(2a_4^p)\frac{2^p}{\Delta^p}\Big[\sum_{k=1}^{p-1}\binom{p}{k}(4a_4^2)^{p-k}\sum_{i=0}^{k}\binom{k}{i}(6a_4)^i(-9a_6)^{k-i}\mu_{p+k+i,3p}$$
$$+\sum_{k=1}^{p-1}\binom{p}{k}(6a_4)^k(-9a_6)^{p-k}\mu_{2p+k,3p}\Big]\Bigg),$$

H_1 is

$$\frac{-1}{p}\Bigg(9\delta(a_4)\,a_4^p(3^p)\gamma_{3p,p}$$
$$-\frac{9}{2}a_4^p\Big(2(3a_6)^p\mu_{2p,3p}+2(3^p)\mu_{5p,3p}+2(1+2^p)a_4^p\mu_{3p,3p}+(2-8^p)\mu_{2p,p}\Big)$$
$$+(9a_6^p-\frac{3}{2}a_4^p)\Big(2(3^p)\mu_{4p,3p}+2(1+2^p)a_4^p\mu_{2p,3p}+(2-8^p)\mu_{p,p}\Big)$$
$$-a_4^{2p}\Big(2(3^p)\mu_{3p,3p}+(2-8^p)\mu_{0,p}\Big)$$
$$+\Big(\Big(\frac{3}{2}\,a_4^p-9\,a_6^p\Big)\,\delta(a_4)+\frac{9}{2}\,a_4^p\,\delta(a_6)\Big)\Big(2(3^p)\gamma_{5p,3p}+(2-8^p)\gamma_{2p,p}\Big)$$
$$+\Big(-3a_4^{2p}+\frac{9}{2}\,a_6^pa_4^p\Big)\Big(2(3^p)\mu_{5p,5p}+(2-8^p)\mu_{2p,3p}\Big)\Bigg)/\Delta^p,$$

H_2 is

$$\frac{1}{p}\sum_{k=1}^{p-1}\sum_{i=0}^{k}\sum_{j=0}^{p-i}\binom{p}{k}\binom{k}{i}\binom{p-i}{j}(-1)^{k-i}\left(\frac{9(2^4)}{\Delta}\right)^{p-i}(2a_4)^j(-3a_6)^{p-i-j}$$

$$\times\Bigg[\Big((24a_4^{2p}-36a_6^pa_4^p)\mu_{2p+j,3p+2(p-i)}+(36a_6^pa_4^p+16a_4^{3p}-54a_6^{2p})\mu_{p+j,3p+2(p-i)}$$

$$+24a_6^pa_4^{2p}\mu_{j,3p+2(p-i)}+36\,a_4^p\mu_{2p+j,p+2(p-i)}$$

$$+(12\,a_4^p-72\,a_6^p)\mu_{p+j,p+2(p-i)}+8a_4^{2p}\mu_{j,p+2(p-i)}\Big)\Big/\Delta^p$$

$$+\Big(36a_4^p\,\delta(a_4)\gamma_{3p+j,p+2(p-i)}$$

$$+\big((72\,a_6^p-12\,a_4^p)\,\delta(a_4)-36\,a_4^p\,\delta(a_6)\big)\gamma_{2p+j,p+2(p-i)}$$

$$+\big(28a_4^{2p}\delta(a_4)+(72a_6^p-12a_4^p)\,\delta(a_6)\big)\gamma_{p+j,p+2(p-i)}$$

$$+\big((72a_4^pa_6^p-24a_4^{2p})\delta(a_4)-8a_4^{2p}\delta(a_6)\big)\gamma_{j,p+2(p-i)}\Big)\Big/\Delta^p$$

$$+\Big(\big((54a_6^{2p}-16a_4^{3p}-36a_4^pa_6^p)\delta(a_4)+(36a_4^pa_6^p-24a_4^{2p})\,\delta(a_6)\big)\gamma_{2p+j,3p+2(p-i)}$$

$$+\big((24a_4^{3p}-60a_4^{2p}a_6^p)\,\delta(a_4)$$

$$+(54a_6^{2p}-16a_4^{3p}-36a_4^pa_6^p)\,\delta(a_6)\big)\gamma_{p+j,3p+2(p-i)}\Big)\Big/\Delta^p\Bigg],$$

H_3 is

$$-\frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k}3^ka_4^{p-k}$$

$$\times\Bigg(\Big((2304\,a_4^{3p}+10368\,a_6^{2p}-8640\,a_6^p\,a_4^p+1152\,a_4^{2p})\,\mu_{2p+2k,p}$$

$$+(-1920\,a_4^{3p}-576\,a_6^p\,a_4^{2p})\,\mu_{p+2k,p}$$

$$+(512\,a_4^{4p}+10368\,a_6^{2p}\,a_4^p-10368\,a_6^p\,a_4^{2p}+2304\,a_4^{3p})\,\mu_{2k,p}$$

$$+(-10368\,a_6^{2p}\,a_4^p+3456\,a_6^p\,a_4^{2p}$$

$$+3072\,a_4^{4p}+7776\,a_6^{3p}-4608\,a_6^p\,a_4^{3p})\,\mu_{2p+2k,3p}$$

$$+(11520\,a_6^p\,a_4^{3p}+1024\,a_4^{5p}-12096\,a_6^{2p}\,a_4^{2p}-2304\,a_4^{4p})\,\mu_{p+2k,3p}$$

$$+(-5184\,a_6^{3p}\,a_4^p-2304a_6^pa_4^{3p}+1536a_6^pa_4^{4p}+6912a_6^{2p}a_4^{2p})\mu_{2k,3p}\Big)\Big/\Delta^{2p}$$

$$+\Big(\big((1920a_4^{3p}+576a_6^pa_4^{2p})\,\delta(a_4)$$

$$+(8640a_6^pa_4^p-1152a_4^{2p}-2304a_4^{3p}-10368a_6^{2p})\,\delta(a_6)\big)\,\gamma_{2p+2k,p}$$

$$+\big((1792\,a_4^{4p}-1152\,a_4^{3p}+1728\,a_6^p\,a_4^{2p})\,\delta(a_4)$$

$$+(1920\,a_4^{3p}+576\,a_6^p\,a_4^{2p})\,\delta(a_6)\big)\gamma_{p+2k,p}$$

$$+\big((6912\,a_6^p\,a_4^{3p}+1728\,a_6^{2p}\,a_4^p$$

$$+2592\,a_6^{3p}-3072\,a_4^{4p}-2304\,a_6^p\,a_4^{2p})\,\delta(a_4)$$

$$+(-512\,a_4^{4p}-10368\,a_6^{2p}\,a_4^p+10368\,a_6^p\,a_4^{2p}-2304\,a_4^{3p})\,\delta(a_6)\big)\gamma_{2k,p}\Big)\Big/\Delta^{2p}\Bigg),$$

H_4 *is*

$$
\begin{aligned}
&\Big(-\frac{1}{2}\,\big(-3732480\,a_6^{2p}\,a_4^{2p} - 829440\,a_6^{p}\,a_4^{4p} \\
&\quad + 3359232\,a_6^{3p}\,a_4^{p} + 3456\,a_4^{2p}\,\Delta^p + 774144\,a_4^{5p} \\
&\quad + 1990656\,a_6^{p}\,a_4^{3p} - 12960\,a_4^{p}\,a_6^{p}\,\Delta^p - 663552\,a_4^{4p}\big)\tau_{2p,p} \\
&\quad - \frac{1}{2}\,\big(2592\,a_4^{2p}\,\Delta^p - 3317760\,a_4^{5p} - 7713792\,a_6^{2p}\,a_4^{3p} \\
&\quad + 19440\,a_6^{2p}\,\Delta^p + 9123840\,a_6^{p}\,a_4^{4p} - 7278336\,a_6^{4p} \\
&\quad + 2985984\,a_6^{2p}\,a_4^{2p} - 1327104\,a_6^{p}\,a_4^{3p} + 3359232\,a_6^{3p}\,a_4^{p} \\
&\quad + 24576\,a_4^{6p} - 12960\,a_4^{p}\,a_6^{p}\,\Delta^p - 288\,a_4^{3p}\,\Delta^p\big)\tau_{p,p} \\
&\quad - \frac{1}{2}\,\big(-1769472\,a_4^{6p} - 8957952\,a_6^{3p}\,a_4^{2p} - 11232\,a_4^{2p}\,a_6^{p}\,\Delta^p \\
&\quad + 2875392\,a_4^{5p}\,a_6^{p} + 3456\,a_4^{3p}\,\Delta^p + 1327104\,a_4^{5p} \\
&\quad + 20404224\,a_6^{2p}\,a_4^{3p} - 10616832\,a_6^{p}\,a_4^{4p}\big)\tau_{0,p}\Big) \Big/ \big(\Delta^{3p}\big) \\
&+ \Big(-\frac{1}{2}\,\big(-393216\,a_4^{7p} + 22394880\,a_6^{3p}\,a_4^{2p} - 3456\,a_4^{2p}\,a_6^{p}\,\Delta^p \\
&\quad - 11197440\,a_6^{4p}\,a_4^{p} - 10616832\,a_4^{5p}\,a_6^{p} + 2592\,a_4^{p}\,a_6^{2p}\,\Delta^p \\
&\quad + 3317760\,a_6^{p}\,a_4^{4p} + 1152\,a_4^{4p}\,\Delta^p + 2654208\,a_4^{6p} \\
&\quad - 14929920\,a_6^{2p}\,a_4^{3p} + 9953280\,a_6^{2p}\,a_4^{4p} + 1152\,a_4^{3p}\,\Delta^p\big)\tau_{2p,3p} \\
&\quad - \frac{1}{2}\,\big(12607488\,a_4^{5p}\,a_6^{p} - 8957952\,a_6^{3p}\,a_4^{2p} - 3244032\,a_6^{p}\,a_4^{6p} \\
&\quad + 19408896\,a_6^{3p}\,a_4^{3p} + 1536\,a_4^{4p}\,\Delta^p + 13436928\,a_6^{4p}\,a_4^{p} \\
&\quad - 28864512\,a_6^{2p}\,a_4^{4p} + 5184\,a_4^{2p}\,a_6^{p}\,\Delta^p + 11664\,a_6^{3p}\,\Delta^p \\
&\quad - 1327104\,a_4^{6p} - 15552\,a_4^{p}\,a_6^{2p}\,\Delta^p + 1769472\,a_4^{7p} \\
&\quad - 2304\,a_4^{3p}\,a_6^{p}\,\Delta^p + 1990656\,a_6^{2p}\,a_4^{3p} - 6718464\,a_6^{5p}\big)\tau_{p,3p} \\
&\quad - \frac{1}{2}\,\big(-20901888\,a_6^{3p}\,a_4^{3p} - 18144\,a_4^{2p}\,a_6^{2p}\,\Delta^p - 3456\,a_4^{4p}\,\Delta^p \\
&\quad + 9953280\,a_6^{2p}\,a_4^{4p} + 13436928\,a_6^{4p}\,a_4^{2p} + 17280\,a_4^{3p}\,a_6^{p}\,\Delta^p \\
&\quad + 1769472\,a_6^{p}\,a_4^{6p} + 1664\,a_4^{5p}\,\Delta^p \\
&\quad - 2654208\,a_6^{2p}\,a_4^{5p} - 1327104\,a_4^{5p}\,a_6^{p}\big)\tau_{0,3p}\Big) \Big/ \big(\Delta^{3p}\big) \\
&+ \Big(-\frac{1}{2}\,\big(-10368\,a_4^{3p}\,a_6^{p}\,\Delta^p + 31104\,a_4^{2p}\,a_6^{2p}\,\Delta^p - 54\,\Delta^{2p}\,a_4^{p} \\
&\quad + 10368\,\Delta^p\,a_6^{p}\,a_4^{4p} - 23328\,\Delta^p\,a_6^{3p}\,a_4^{p} - 4608\,a_4^{5p}\,\Delta^p\big)\tau_{2p,5p}\Big) \Big/ \big(\Delta^{3p}\big),
\end{aligned}
$$

H_5 *is*

$$\Big(\Big(\frac{1}{2}\big(- 14556672\, a_6^{4p} - 6635520\, a_4^{5p} + 6718464\, a_6^{3p}\, a_4^{p} + 5971968\, a_6^{2p}\, a_4^{2p} + 2880\, a_4^{2p}\, \Delta^p + 18144\, a_6^{2p}\, \Delta^p - 5184\, a_4^{3p}\, \Delta^p - 15427584\, a_6^{2p}\, a_4^{3p} - 2654208\, a_6^{p}\, a_4^{3p} - 8640\, a_4^{p}\, a_6^{p}\, \Delta^p + 49152\, a_4^{6p} + 18247680\, a_6^{p}\, a_4^{4p}\big)\delta(a_4) + \frac{1}{2}\big(1548288\, a_4^{5p} + 6912\, a_4^{2p}\, \Delta^p - 25920\, a_4^{p}\, a_6^{p}\, \Delta^p - 1658880\, a_6^{p}\, a_4^{4p} + 6718464 a_6^{3p} a_4^{p} - 7464960 a_6^{2p} a_4^{2p} + 3981312 a_6^{p} a_4^{3p} - 1327104 a_4^{4p}\big)\delta(a_6) + \frac{1}{2}\big(3584\, a_4^{4p}\, \delta(3)\, \Delta^p + 3456\, \delta(3)\, a_4^{2p}\, a_6^{p}\, \Delta^p + 4608\, \delta(2)\, a_4^{4p}\, \Delta^p + 18\, \delta(3)\, a_4^{p}\, \Delta^{2p} - 2304\, a_4^{3p}\, \delta(3)\, \Delta^p\big)\Big)\mu_{2p,p}\Big)\Big/\big(\Delta^{3p}\big),$$

H_6 *is*

$$\Big(\Big(\frac{1}{2}\big(3840\, a_4^{3p}\, \Delta^p - 25214976\, a_6^{p}\, a_4^{4p} + 3981312\, a_4^{5p} + 7409664\, a_4^{5p}\, a_6^{p} + 4608\, a_4^{2p}\, a_6^{p}\, \Delta^p - 5087232\, a_4^{6p} + 48273408\, a_6^{2p}\, a_4^{3p} - 24634368\, a_6^{3p}\, a_4^{2p}\big)\delta(a_4) + \frac{1}{2}\big(- 14556672\, a_6^{4p} - 15427584\, a_6^{2p}\, a_4^{3p} - 25920\, a_4^{p}\, a_6^{p}\, \Delta^p - 2654208\, a_6^{p}\, a_4^{3p} - 6635520\, a_4^{5p} + 38880\, a_6^{2p}\, \Delta^p + 18247680\, a_6^{p}\, a_4^{4p} + 5184\, a_4^{2p}\, \Delta^p + 6718464\, a_6^{3p}\, a_4^{p} + 5971968\, a_6^{2p}\, a_4^{2p} - 576\, a_4^{3p}\, \Delta^p + 49152\, a_4^{6p}\big)\delta(a_6) + \frac{1}{2}\big(5184\delta(3) a_6^{3p} \Delta^p - 24\delta(2) a_4^{p} \Delta^{2p} - 36\delta(3) a_6^{p} \Delta^{2p} + 23328\delta(2) a_6^{3p} \Delta^p + 36\, \delta(2)\, a_6^{p}\, \Delta^{2p} - 6912\, \delta(2)\, a_4^{4p}\, \Delta^p + 12672\, a_4^{3p}\, \delta(3)\, a_6^{p}\, \Delta^p - 15552\, \delta(2)\, a_6^{2p}\, a_4^{p}\, \Delta^p + 3456\, \delta(3)\, a_6^{2p}\, a_4^{p}\, \Delta^p + 10368\, \delta(2)\, a_4^{3p}\, a_6^{p}\, \Delta^p - 4608\, \delta(3)\, a_4^{2p}\, a_6^{p}\, \Delta^p - 9984\, a_4^{4p}\, \delta(3)\, \Delta^p + 6\, \delta(3)\, a_4^{p}\, \Delta^{2p}\big)\Big)\mu_{p,p} + \Big(\frac{1}{2}\big(7962624\, a_6^{p}\, a_4^{4p} - 33841152\, a_6^{2p}\, a_4^{3p} - 22781952\, a_4^{5p}\, a_6^{p} + 5308416\, a_4^{6p} + 52254720\, a_6^{3p}\, a_4^{2p} - 29113344\, a_6^{4p}\, a_4^{p} + 54\, \Delta^{2p}\, a_4^{p} - 786432\, a_4^{7p} + 6912\, a_4^{2p}\, a_6^{p}\, \Delta^p + 10368\, a_4^{p}\, a_6^{2p}\, \Delta^p - 2304\, a_4^{3p}\, \Delta^p + 1280\, a_4^{4p}\, \Delta^p + 21565440\, a_6^{2p}\, a_4^{4p}\big)\delta(a_4) + \frac{1}{2}\big(40808448\, a_6^{2p}\, a_4^{3p} - 22464\, a_4^{2p}\, a_6^{p}\, \Delta^p - 17915904\, a_6^{3p}\, a_4^{2p} + 5750784\, a_4^{5p}\, a_6^{p} + 2654208\, a_4^{5p} - 21233664\, a_6^{p}\, a_4^{4p} - 3538944\, a_4^{6p} + 6912\, a_4^{3p}\, \Delta^p\big)\delta(a_6) + \frac{1}{2}\big(- 6912\, \delta(2)\, a_4^{3p}\, a_6^{p}\, \Delta^p - 2048\, a_4^{5p}\, \delta(3)\, \Delta^p + 4608\, a_4^{4p}\, \delta(3)\, \Delta^p - 14\, \delta(3)\, a_4^{2p}\, \Delta^{2p} + 23040\, \delta(3)\, a_6^{2p}\, a_4^{2p}\, \Delta^p - 26880\, a_4^{3p}\, \delta(3)\, a_6^{p}\, \Delta^p + 2\, \delta(2)\, a_4^{2p}\, \Delta^{2p}\big)\Big)\mu_{0,p}\Big)\Big/\big(\Delta^{3p}\big),$$

H_7 *is*

$$
\begin{aligned}
&\Big(\frac{1}{2}\big(-17915904\,a_6^{3p}\,a_4^{2p}+25214976\,a_4^{5p}\,a_6^{p}+7776\,a_6^{3p}\,\Delta^{p}\\
&\qquad+38817792\,a_6^{3p}\,a_4^{3p}+4608\,a_4^{3p}\,a_6^{p}\,\Delta^{p}+3456\,a_4^{2p}\,a_6^{p}\,\Delta^{p}\\
&\qquad+3981312\,a_6^{2p}\,a_4^{3p}+26873856\,a_6^{4p}\,a_4^{p}-2654208\,a_4^{6p}\\
&\qquad-13436928\,a_6^{5p}-30\,\Delta^{2p}\,a_4^{p}-6488064\,a_6^{p}\,a_4^{6p}+3538944\,a_4^{7p}\\
&\qquad-36\Delta^{2p}a_6^{p}-10368a_4^{p}a_6^{2p}\Delta^{p}-3072a_4^{4p}\Delta^{p}-57729024a_6^{2p}a_4^{4p}\big)\delta(a_4)\\
&\quad+\frac{1}{2}\big(54\,\Delta^{2p}\,a_4^{p}+5184\,a_4^{p}\,a_6^{2p}\,\Delta^{p}-21233664\,a_4^{5p}\,a_6^{p}\\
&\qquad+19906560\,a_6^{2p}\,a_4^{4p}+2304\,a_4^{3p}\,\Delta^{p}+6635520\,a_6^{p}\,a_4^{4p}\\
&\qquad-29859840\,a_6^{2p}\,a_4^{3p}+2304\,a_4^{4p}\,\Delta^{p}-6912\,a_4^{2p}\,a_6^{p}\,\Delta^{p}\\
&\qquad+44789760a_6^{3p}a_4^{2p}+5308416a_4^{6p}-22394880a_6^{4p}a_4^{p}-786432a_4^{7p}\big)\delta(a_6)\\
&\quad+\frac{1}{2}\big(25920\,a_6^{3p}\,\delta(3)\,a_4^{p}\,\Delta^{p}+6144\,a_4^{5p}\,\delta(3)\,\Delta^{p}-36\,\delta(2)\,a_4^{p}\,a_6^{p}\,\Delta^{2p}\\
&\qquad+11520\,a_4^{3p}\,\delta(3)\,a_6^{p}\,\Delta^{p}+6\,\delta(2)\,a_4^{2p}\,\Delta^{2p}+6\,\delta(3)\,a_4^{2p}\,\Delta^{2p}\\
&\qquad+18\delta(3)a_6^{p}a_4^{p}\Delta^{2p}-34560\delta(3)a_6^{2p}a_4^{2p}\Delta^{p}-12288a_4^{4p}\delta(3)a_6^{p}\Delta^{p}\big)\Big)\mu_{2p,3p}\Big/\big(\Delta^{3p}\big),
\end{aligned}
$$

and H_8 *is*

$$
\begin{aligned}
&\Big(-\big(-1271808\,a_4^{6p}+995328\,a_4^{5p}+12068352\,a_6^{2p}\,a_4^{3p}\\
&\qquad-6158592\,a_6^{3p}\,a_4^{2p}+1852416\,a_4^{5p}\,a_6^{p}\\
&\qquad-6303744\,a_6^{p}\,a_4^{4p}+1920\,a_4^{3p}\,\Delta^{p}+1440\,a_4^{2p}\,a_6^{p}\,\Delta^{p}\big)\delta(a_4)^2\\
&\quad-\big(9123840\,a_6^{p}\,a_4^{4p}-7278336\,a_6^{4p}+24576\,a_4^{6p}\\
&\qquad-3317760\,a_4^{5p}-7713792\,a_6^{2p}\,a_4^{3p}-2592\,a_4^{3p}\,\Delta^{p}\\
&\qquad+9072\,a_6^{2p}\,\Delta^{p}-4320\,a_4^{p}\,a_6^{p}\,\Delta^{p}-1327104\,a_6^{p}\,a_4^{3p}\\
&\qquad+1440\,a_4^{2p}\,\Delta^{p}+2985984\,a_6^{2p}\,a_4^{2p}+3359232\,a_6^{3p}\,a_4^{p}\big)\delta(a_4)\delta(a_6)\\
&\quad-\big(387072\,a_4^{5p}+1728\,a_4^{2p}\,\Delta^{p}-1866240\,a_6^{2p}\,a_4^{2p}\\
&\qquad+995328\,a_6^{p}\,a_4^{3p}-331776\,a_4^{4p}+1679616\,a_6^{3p}\,a_4^{p}\\
&\qquad\qquad-6480\,a_4^{p}\,a_6^{p}\,\Delta^{p}-414720\,a_6^{p}\,a_4^{4p}\big)\delta(a_6)^2\\
&\quad-\big(18\,\delta(2)\,a_6^{p}\,\Delta^{2p}-18\,\delta(3)\,a_6^{p}\,\Delta^{2p}+1728\,\delta(3)\,a_6^{2p}\,a_4^{p}\,\Delta^{p}\\
&\qquad-7776\,\delta(2)\,a_6^{2p}\,a_4^{p}\,\Delta^{p}+6336\,a_4^{3p}\,\delta(3)\,a_6^{p}\,\Delta^{p}\\
&\qquad-2304\,\delta(3)\,a_4^{2p}\,a_6^{p}\,\Delta^{p}+5184\,\delta(2)\,a_4^{3p}\,a_6^{p}\,\Delta^{p}\\
&\qquad+3\,\delta(3)\,a_4^{p}\,\Delta^{2p}-12\,\delta(2)\,a_4^{p}\,\Delta^{2p}-3456\,\delta(2)\,a_4^{4p}\,\Delta^{p}\\
&\qquad+11664\,\delta(2)\,a_6^{3p}\,\Delta^{p}+2592\,\delta(3)\,a_6^{3p}\,\Delta^{p}-4992\,a_4^{4p}\,\delta(3)\,\Delta^{p}\big)\delta(a_4)\\
&\quad-\big(1792\,a_4^{4p}\,\delta(3)\,\Delta^{p}-1152\,a_4^{3p}\,\delta(3)\,\Delta^{p}+1728\,\delta(3)\,a_4^{2p}\,a_6^{p}\,\Delta^{p}\\
&\qquad\qquad+9\,\delta(3)\,a_4^{p}\,\Delta^{2p}+2304\,\delta(2)\,a_4^{4p}\,\Delta^{p}\big)\delta(a_6)\Big)\frac{\gamma_{2p,p}}{\Delta^{3p}}.
\end{aligned}
$$

Proof. We simply apply the most recent propositions to the actual formulas from Proposition 6.1. It should be noted that the H_i correspond to the F_i in Proposition 6.1 and that upon further analysis certain terms like $\frac{(-1-2^p)x^pD^p}{y^{3p}}$ have zero residue even though it is not immediately obvious that the term is regular. □

From now on, when we refer to H_0, H_1, H_2, H_3, H_4, H_5, H_6, H_7, and H_8 we will mean the polynomials in this theorem. We note that H_0, H_1, H_2, H_3, H_5, H_6, H_7 are in M_1^1 and are linear in $\delta(a_4)$ and $\delta(a_6)$, $H_4 \in M_1^0$, and $H_8 \in M_1^1$ is quadratic in $\delta(a_4)$ and $\delta(a_6)$.

7. Order two modular forms

We remind ourselves that ϕ, the unique lifting of the Frobenius morphism to R, extends to a homomorphism from $M_1^1 \to M_1^2$ by taking, e.g., $a_4 \mapsto a_4^p + p\delta(a_4)$ and $\delta(a_6) \mapsto \delta(a_6)^p + p\delta(a_6)$. Hence if we start with a polynomial in M^0 like $\gamma_{a,b}$, then $\phi(\gamma_{a,b}) = \gamma_{a,b}(a_4^p + p\delta(a_4), a_6^p + p\delta(a_6)) \in M^1$, where by this notation we mean substitute $a_4^p + p\delta(a_4)$ in for a_4 and $a_6^p + p\delta(a_6)$ in for a_6.

Definition 7.1. Let $\widetilde{\gamma}_{a,b}$ be the polynomial in M^1 such that $\phi(\gamma_{a,b}) = \gamma_{a,b}(a_4^p, a_6^p) + p\widetilde{\gamma}_{a,b}$.

An explicit formula for $\widetilde{\gamma}_{a,b}$ is simple to compute by expanding the formula

$$\phi(\gamma_{a,b}) = \begin{cases} 0 & \text{if b is even,} \\ \sum_{k=0}^{\infty} \binom{m+k}{3k+2-n} \binom{m-2k-2+n}{\frac{b-1}{2}} & \\ \quad (-1)^{m+k-\frac{b-1}{2}} (a_4^p + p\delta(a_4))^{3k+2-n} (a_6^p + p\delta(a_6))^{m-2k-2+n-\frac{b-1}{2}} & \text{if b is odd.} \end{cases}$$

and modulo p^2, $\widetilde{\gamma}_{a,b}$ is linear in $\delta(a_4), \delta(a_6)$. In addition $\phi(\gamma_{a,b}) = \gamma_{a,b}^p + p\delta(\gamma_{a,b})$; however, note that $\widetilde{\gamma}_{a,b}$ does not equal $p\delta(\gamma_{a,b})$ because the latter is missing the terms from $\gamma_{a,b}^p$ whose coefficients are divisible by p.

Definition 7.2. Let $\widetilde{\mu}_{a,b}$ be the polynomial in M^1 such that $\phi(\mu_{a,b}) = \mu_{a,b}(a_4^p, a_6^p) + p\widetilde{\mu}_{a,b}$.

We recall that the isogeny covariant differential modular form $f_{\text{jet}}h_{\text{jet}}$ is $\phi(f_{\text{jet}})$.

Theorem 7.3. *The reduction modulo p^2 of $f_{\text{jet}}h_{\text{jet}}$ is*

$$\begin{aligned} &\left[\frac{-72a_6^{p^2}\delta(a_4)^p + 48a_4^{p^2}\delta(a_6)^p}{\Delta^{p^2}}\right]\gamma_{2p,p}(a_4^p, a_6^p) \\ &\quad + \frac{1}{\Delta^{p^2}}\Big[-8a_4^{2p^2}\mu_{0,p}(a_4^p, a_6^p) + 72a_6^{p^2}\mu_{p,p}(a_4^p, a_6^p) \\ &\quad - 48a_4^{p^2}\mu_{2p,p}(a_4^p, a_6^p) - 24a_4^{2p^2}\mu_{3p,3p}(a_4^p, a_6^p)\Big] \\ &\quad + p\left[\frac{-72a_6^{p^2}\delta^2(a_4) + 48a_4^{p^2}\delta^2(a_6)}{\Delta^{p^2}}\right]\gamma_{2p,p}(a_4^p, a_6^p) + pJ_0, \end{aligned}$$

where J_0 *is*

$$\Big(\frac{1}{\Delta^{p^2}}\Big[(-72a_6^{p^2}\delta(a_4)^p + 48a_4^{p^2}\delta(a_6)^p)\widetilde{\gamma}_{2p,p} - 8a_4^{2p^2}\widetilde{\mu}_{0,p} + 72a_6^{p^2}\widetilde{\mu}_{p,p} - 48a_4^{p^2}\widetilde{\mu}_{2p,p} - 24a_4^{2p^2}\widetilde{\mu}_{3p,3p} + \big((27\delta(2)+66\delta(3))a_6^{p^2}\delta(a_4)^p + (-28\delta(3)-42\delta(2))a_4^{p^2}\delta(a_6)^p\big)\gamma_{2p,p}^p + 20\delta(2)^p a_4^{2p^2}\mu_{0,p}^p + (-27\delta(2)-66\delta(3))a_6^{p^2}\mu_{p,p}^p + (42\delta(2)+28\delta(3))a_4^{p^2}\mu_{2p,p}^p + 18\delta(2)a_4^{p^2}a_6^{p^2}\mu_{2p,3p}^p + (8\delta(3)+24\delta(2))a_4^{2p^2}\mu_{3p,3p}^p\Big] + H_0^p + H_1^p + H_2^p + H_3^p + H_4^p + H_5^p + H_6^p + H_7^p + H_8^p\Big).$$

Proof. Let H_i be the polynomials from Theorem 6.11. The formula follows immediately from the fact that $f_{\text{jet}}h_{\text{jet}} = \phi(f_{\text{jet}})$. □

Next working modulo p^2, $h_{\text{jet}} = (f_{\text{jet}}h_{\text{jet}})/f_{\text{jet}}$ is $\frac{h_0}{f_0} + p\left(\frac{-h_0f_1}{f_0^2} + \frac{h_1}{f_0}\right)$, where f_0 is the coefficient of p^0 in f_{jet}, f_1 is the coefficient of p, h_0 is the coefficient of p^0 in $f_{\text{jet}}h_{\text{jet}}$, and h_1 is the coefficient of p. In particular

$$f_0 = \left[\frac{-72a_6^p\delta(a_4) + 48a_4^p\delta(a_6)}{\Delta^p}\right]\gamma_{2p,p} + \frac{1}{\Delta^p}\Big[-8a_4^{2p}\mu_{0,p} + 72a_6^p\mu_{p,p} - 48a_4^p\mu_{2p,p} - 24a_4^{2p}\mu_{3p,3p}\Big],$$

$$f_1 = \frac{1}{\Delta^p}\Big[\big((27\delta(2)+66\delta(3))a_6^p\delta(a_4) + (-28\delta(3)-42\delta(2))a_4^p\delta(a_6)\big)\gamma_{2p,p} + 20\delta(2)a_4^{2p}\mu_{0,p} + (-27\delta(2)-66\delta(3))a_6^p\mu_{p,p} + (42\delta(2)+28\delta(3))a_4^p\mu_{2p,p} + 18\delta(2)a_4^pa_6^p\mu_{2p,3p} + (8\delta(3)+24\delta(2))a_4^{2p}\mu_{3p,3p}\Big] + \Big(H_0 + H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + H_8\Big),$$

$$h_0 = \left[\frac{-72a_6^{p^2}\delta(a_4)^p + 48a_4^{p^2}\delta(a_6)^p}{\Delta^{p^2}}\right]\gamma_{2p,p}(a_4^p, a_6^p) + \frac{1}{\Delta^{p^2}}\Big[-8a_4^{2p^2}\mu_{0,p}(a_4^p, a_6^p) + 72a_6^{p^2}\mu_{p,p}(a_4^p, a_6^p) - 48a_4^{p^2}\mu_{2p,p}(a_4^p, a_6^p) - 24a_4^{2p^2}\mu_{3p,3p}(a_4^p, a_6^p)\Big],$$

$$h_1 = \left[\frac{-72a_6^{p^2}\delta^2(a_4) + 48a_4^{p^2}\delta^2(a_6)}{\Delta^{p^2}}\right]\gamma_{2p,p}(a_4^p, a_6^p) + J_0.$$

Therefore we have the following explicit formulation for h_{jet} where J_0 is the polynomial from Theorem 7.3.

Theorem 7.4. *The reduction modulo p^2 of h_{jet} is*

$$\left(\big(-72a_6^{p^2}\delta(a_4)^p+48a_4^{p^2}\delta(a_6)^p\big)\gamma_{2p,p}(a_4^p,a_6^p)-8a_4^{2p^2}\mu_{0,p}(a_4^p,a_6^p)\right.$$
$$\left.+72a_6^{p^2}\mu_{p,p}(a_4^p,a_6^p)-48a_4^{p^2}\mu_{2p,p}(a_4^p,a_6^p)-24a_4^{2p^2}\mu_{3p,3p}(a_4^p,a_6^p)\right)\Big/$$
$$\left(\Delta^{p^2-p}\Big(\big(-72a_6^p\delta(a_4)+48a_4^p\delta(a_6)\big)\gamma_{2p,p}\right.$$
$$\left.-8a_4^{2p}\mu_{0,p}+72a_6^p\mu_{p,p}-48a_4^p\mu_{2p,p}-24a_4^{2p}\mu_{3p,3p}\Big)\right)+pK_0$$
$$+\frac{p\big(-72a_6^{p^2}\delta^2(a_4)+48a_4^{p^2}\delta^2(a_6)\big)\gamma_{2p,p}(a_4^p,a_6^p)}{\Delta^{p^2-p}\Big(\big(-72a_6^p\delta(a_4)+48a_4^p\delta(a_6)\big)\gamma_{2p,p}-8a_4^{2p}\mu_{0,p}+72a_6^p\mu_{p,p}-48a_4^p\mu_{2p,p}-24a_4^{2p}\mu_{3p,3p}\Big)},$$

where K_0 is

$$\frac{\Delta^pJ_0}{\Big(\big(-72a_6^p\delta(a_4)+48a_4^p\delta(a_6)\big)\gamma_{2p,p}-8a_4^{2p}\mu_{0,p}+72a_6^p\mu_{p,p}-48a_4^p\mu_{2p,p}-24a_4^{2p}\mu_{3p,3p}\Big)}$$
$$-\left(\big(-72a_6^{p^2}\delta(a_4)^p+48a_4^{p^2}\delta(a_6)^p\big)\gamma_{2p,p}(a_4^p,a_6^p)-8a_4^{2p^2}\mu_{0,p}(a_4^p,a_6^p)\right.$$
$$\left.+72a_6^{p^2}\mu_{p,p}(a_4^p,a_6^p)-48a_4^{p^2}\mu_{2p,p}(a_4^p,a_6^p)-24a_4^{2p^2}\mu_{3p,3p}(a_4^p,a_6^p)\right)$$
$$\times\left(\frac{1}{\Delta^p}\Big[\big((27\delta(2)+66\delta(3))a_6^p\delta(a_4)+(-28\delta(3)-42\delta(2))a_4^p\delta(a_6)\big)\gamma_{2p,p}\right.$$
$$+20\delta(2)a_4^{2p}\mu_{0,p}+(-27\delta(2)-66\delta(3))a_6^p\mu_{p,p}+(42\delta(2)+28\delta(3))a_4^p\mu_{2p,p}$$
$$\left.+18\delta(2)a_4^pa_6^p\mu_{2p,3p}+(8\delta(3)+24\delta(2))a_4^{2p}\mu_{3p,3p}\Big]+\sum_{i=0}^{8}H_i\right)\Big/$$
$$\left(\Big(\big(-72a_6^p\delta(a_4)+48a_4^p\delta(a_6)\big)\gamma_{2p,p}\right.$$
$$\left.-8a_4^{2p}\mu_{0,p}+72a_6^p\mu_{p,p}-48a_4^p\mu_{2p,p}-24a_4^{2p}\mu_{3p,3p}\Big)^2\Delta^{p^2-2p}\right).$$

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