FURTHER NONEXISTENCE OF FIBONACCI TRIANGLES

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ABSTRACT. A Fibonacci triangle is a triangle with integer area and sides whose lengths are Fibonacci numbers. An example of a Fibonacci triangle is the triangle whose sides have lengths $(5, 5, 8)$. This example, first given by H. Harborth and A. Kemnitz in a paper where they proved that any Fibonacci triangle must be isosceles and any other Fibonacci triangle must have side lengths (F_{n-k}, F_n, F_n) for $2 \leq k < n$ [1], is the only known example of a Fibonacci triangle. Later H. Harborth, A Kemnitz, and N. Robbins proved that there are no Fibonacci triangles with side lengths (F_{n-k}, F_n, F_n) and$1 \leq k \leq 5$ [2], and have conjectured that there are no other Fibonacci triangles besides the triangle with sides $(5, 5, 8)$. In what follows we demonstrate the non-existence of Fibonacci triangles for a number of values of k.

1. INTRODUCTION

Fibonacci trianges, first introduced by H. Harborth and A. Kemnitz in [1], are triangles whose side lengths are Fibonacci numbers and whose area is integral. In general, a triangle with integer side lengths and integer area is referred to as a Heron triangle. In particular a Heron triangle whose side lengths are Fibonacci numbers is called a Fibonacci triangle. At this time there is only one known example of a Fibonacci triangle, namely $(5, 5, 8)$. It was proved in [1] that $(5, 5, 8)$ is the only Fibonacci triangle of the type (F_k, F_k, F_n) for $1 \leq k \leq n$. In the same paper they also proved that there are no Fibonacci triangles of the type (F_{n-1}, F_n, F_n) and conjecture, "Perhaps $(5, 5, 8)$ is the unique Fibonacci triangle?" In a later paper H. Harborth, A Kemnitz, and N. Robbins prove that there are no Fibonacci triangles with side lengths (F_{n-k}, F_n, F_n) and$1 \leq k \leq 5$ [2].

We provide further evidence supporting the nonexistence conjecture by proving the following theorem.

Theorem 1.1. *There are no Fibonacci triangles with side lengths* (F_{n-k}, F_n, F_n) *and* $6 \leq k \leq 10$.

The techniques below used to prove nonexistence are different than those used in the previous references. They rely among other things on the periodicity of the Fibonacci numbers modulo m and computations using to Jacobi symbol. We remind the reader in the first section about various properties of the Fibonacci numbers that we will use and provide a brief outline of the Jacobi symbol. Then we discuss requirements necessary for a triangle to be a Fibonacci triangle which leads use to develop some general purpose tools to use in proving nonexistence of Fibonacci triangles. In the final section we prove the nonexistence on Fibonacci triangles for $6 \leq k \leq 10$ on a case by case basis for each value of k, proving the theorem in the introduction.

2. Properties of Fibonacci Numbers

The Fibonacci numbers are the numbers in a sequence of numbers F_i where $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 3$. The numbers in this sequence have many interesting properties including the following identities:

$$F_{n+m} = F_m F_{n+1} + F_{m-1} F_n \tag{2.1}$$

$$F_{2n-1} = F_n^2 + F_{n-1}^2 \tag{2.2}$$

$$F_{2n} = F_n^2 + 2F_{n-1}F_n \tag{2.3}$$

for m and n positive integers. When considered modulo F_n,

$$F_{mn+r} \equiv \begin{cases} F_r, & \text{if } m \bmod 4 = 0; \\ (-1)^{r+1} F_{n-r}, & \text{if } m \bmod 4 = 1; \\ (-1)^n F_r, & \text{if } m \bmod 4 = 2; \\ (-1)^{r+1+n} F_{n-r}, & \text{if } m \bmod 4 = 3. \end{cases}$$

Moreover given two Fibonacci numbers F_n and F_m, the $\gcd(F_n, F_m) = F_{\gcd(n,m)}$ where gcd refers to the greatest common divisor of F_n and F_m. This in combination with the fact that F_{mk} is a multiple of F_m, means that if $m > 2$, then $F_m | F_n$ if and only if $m | n$. To see the if direction of this iff note if $F_m | F_n$, then $F_m = \gcd(F_m, F_n) = F_{\gcd(m,n)}$. When $m > 2$, the Fibonacci numbers are an injective mapping to the integers and so $m = \gcd(m, n)$ implying that $m | n$.

Next we introduce the Jacobi symbol as a tool for proving that a number is not a perfect square. Recall that to define the Jacobi symbol, we must first define the Legendre symbol.

Definition 2.1. Let p be a prime number. Then the symbol $\left(\frac{a}{p}\right)$ will have value 1 if a is a quadratic residue mod p, -1 if a is a quadratic nonresidue mod p, and zero if $p | a$. $\left(\frac{a}{p}\right)$ is called the Legendre symbol.

Then the Jacobi symbol, also denoted $\left(\frac{a}{b}\right)$, is an extension of the Legendre symbol where if b is a prime number $\left(\frac{a}{b}\right)$ is the Legendre symbol as just defined. Otherwise,

Definition 2.2. Let b be an odd, positive integer and a any integer. Let $b = p_1 p_2 \dots p_m$, where the p_i are (not necessarily distinct) primes. The symbol $\left(\frac{a}{b}\right)$ defined by

$$\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right)\cdots\left(\frac{a}{p_m}\right)$$

is called the Jacobi symbol.

So if $\left(\frac{a}{b}\right) = -1$ then a is a quadratic nonresidue modulo b and hence a cannot be a perfect square. The following proposition provides tools for occasionally computing the Jacobi symbol.

Proposition 2.3. *Let b, b_1, and b_2 be odd positive integers. Then*

(i) $\left(\frac{a_1}{b}\right) = \left(\frac{a_2}{b}\right)$ *if $a_1 \equiv a_2$ modulo b;*

(ii) $\left(\frac{a_1 a_2}{b}\right) = \left(\frac{a_1}{b}\right)\left(\frac{a_2}{b}\right)$;

(iii) $\left(\frac{a}{b_1 b_2}\right) = \left(\frac{a}{b_1}\right)\left(\frac{a}{b_2}\right)$.

Furthermore if both a and b are odd positive integers with either $a \equiv 1$ modulo 4 or $b \equiv 1$ modulo 4 and $\gcd(a,b) = 1$, then $\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right)$. Using the Jacobi symbol, following proposition about Fibonacci numbers that implies not only is F_{2^l-1} not a perfect square, but also any number congruent to F_{2^l-1} modulo F_{2^l} is not a perfect square.

Proposition 2.4. *If $l \geq 2$, then the Jacobi symbol*

$$\left(\frac{F_{2^l-1}}{F_{2^l}}\right) = -1.$$

Proof. Using the Fibonacci identities 2.2 and 2.3 we can compute as follows

$$\begin{aligned}
\left(\frac{F_{2^l-1}}{F_{2^l}}\right) &= \left(\frac{F_{2^l-1}}{F_{2^{l-1}}(F_{2^{l-1}} + 2F_{2^{l-1}-1})}\right) \\
&= \left(\frac{F_{2^l-1}}{F_{2^{l-1}}}\right)\left(\frac{F_{2^l-1}}{(F_{2^{l-1}} + 2F_{2^{l-1}-1})}\right) \\
&= \left(\frac{F^2_{2^{l-1}} + F^2_{2^{l-1}-1}}{F_{2^{l-1}}}\right)\left(\frac{F^2_{2^{l-1}} + F^2_{2^{l-1}-1}}{(F_{2^{l-1}} + 2F_{2^{l-1}-1})}\right) \\
&= \left(\frac{F^2_{2^{l-1}} + F^2_{2^{l-1}-1}}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right).
\end{aligned}$$

Then using congruences inside the Jacobi symbol

$$\begin{aligned}
\left(\frac{F_{2^l-1}}{F_{2^l}}\right) &= \left(\frac{F_{2^{l-1}}^2 + F_{2^{l-1}-1}^2 - F_{2^{l-1}}(F_{2^{l-1}} + 2F_{2^{l-1}-1})}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right) \\
&= \left(\frac{F_{2^{l-1}-1}^2 - F_{2^{l-1}}2F_{2^{l-1}-1}}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right) \\
&= \left(\frac{F_{2^{l-1}-1}(F_{2^{l-1}-1} - 2F_{2^{l-1}})}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right) \\
&= \left(\frac{F_{2^{l-1}-1}}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right)\left(\frac{F_{2^{l-1}-1} - 2F_{2^{l-1}}}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right) \\
&= \left(\frac{F_{2^{l-1}-1}}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right)\left(\frac{F_{2^{l-1}-1} - 2F_{2^{l-1}} + 2(F_{2^{l-1}} + 2F_{2^{l-1}-1})}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right) \\
&= \left(\frac{F_{2^{l-1}-1}}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right)\left(\frac{5F_{2^{l-1}-1}}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right) \\
&= \left(\frac{5}{F_{2^{l-1}} + 2F_{2^{l-1}-1}}\right).
\end{aligned}$$

Finally because $5 \equiv 1 \mod 4$ this becomes

$$\left(\frac{F_{2^l-1}}{F_{2^l}}\right) = \left(\frac{F_{2^{l-1}} + 2F_{2^{l-1}-1}}{5}\right).$$

We can now compute the last Jacobi symbol by computing $F_{2^{l-1}} + 2F_{2^{l-1}-1}$ modulo 5. We note that F_n modulo 5 repeats every 20 and so the value of $F_{2^{l-1}}$ modulo 5 is just the value of $F_{2^{l-1} \mod 20}$. If $l = 2$, then $F_{2^{l-1}} + 2F_{2^{l-1}-1} = F_2 + 2F_1 = 3$ which is a nonresidue modulo 5. If $l > 2$, then 2^{l-1} modulo 20 is one of 4, 8, 16, or 12. In each of these cases

- **$2^{l-1} \equiv 4 \mod 20$:** In this case $F_{2^{l-1}} \equiv 3 \mod 5$ and $F_{2^{l-1}-1} \equiv 2 \mod 5$. Therefore $F_{2^{l-1}} + 2F_{2^{l-1}-1} \equiv 3 + 2(2) = 7 \mod 5$ which is a nonresidue modulo 5.
- **$2^{l-1} \equiv 8 \mod 20$:** In this case $F_{2^{l-1}} \equiv 1 \mod 5$ and $F_{2^{l-1}-1} \equiv 3 \mod 5$. Therefore $F_{2^{l-1}} + 2F_{2^{l-1}-1} \equiv 1 + 2(3) = 7 \mod 5$ which is a nonresidue modulo 5.
- **$2^{l-1} \equiv 12 \mod 20$:** In this case $F_{2^{l-1}} \equiv 4 \mod 5$ and $F_{2^{l-1}-1} \equiv 4 \mod 5$. Therefore $F_{2^{l-1}} + 2F_{2^{l-1}-1} \equiv 4 + 2(4) = 12 \mod 5$ which is a nonresidue modulo 5.

- $2^{l-1} \equiv 16 \mod 20$: In this case $F_{2^{l-1}} \equiv 1 \mod 2$ and $F_{2^{l-1}-1} \equiv 0 \mod 5$. Therefore $F_{2^{l-1}} + 2F_{2^{l-1}-1} \equiv 2 + 2(0) = 2 \mod 5$ which is a nonresidue modulo 5.

Since all possible cases result in a quadratic nonresidue, it follows that

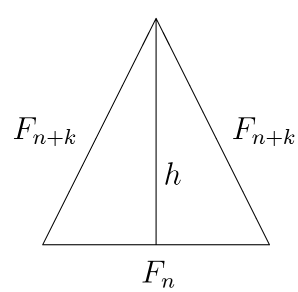
$$\left(\frac{F_{2^l-1}}{F_{2^l}}\right) = \left(\frac{F_{2^{l-1}} + 2F_{2^{l-1}-1}}{5}\right) = -1.$$

□

3. Requirements to be a Fibonacci Triangle

Recall that a Fibonacci triangle is a triangle whose side lengths are Fibonacci numbers and whose area is integral. In [1], H. Harborth and A. Kemnitz showed that any Fibonacci triangle must be isosceles. They also proved that aside from the triangle with sides $(5, 5, 8)$, any other Fibonacci triangle must have side lengths (F_{n-k}, F_n, F_n) where $n > k$ and k represents the difference in index between the short side length and the longer side lengths. Therefore, for a fixed value of k, proving the non-existence of Fibonacci triangles with side lengths (F_{n-k}, F_n, F_n) for all $n > k$ is equivalent to proving the non-existence of Fibonacci triangles with side lengths (F_{n+k}, F_{n+k}, F_n) for all positive integers n. We will use the latter indexing system for possible Fibonacci triangles and from now on all triangles will be referred to by listing the lengths of their sides, albeit without any particular regard to order.

Consider the triangle



This triangle is a Fibonacci triangle when and only when the area of this triangle, $\frac{1}{2}hF_n$, is integral. Therefore we will assume that $\frac{1}{2}hF_n$ is integral and prove the following consequences.

Proposition 3.1. *If (F_{n+k}, F_{n+k}, F_n) is a Fibonacci triangle, then F_n is even.*

Proof. Let A be the area of (F_{n+k}, F_{n+k}, F_n) and suppose A is an integer. Then $16A^2 = 4F_n^2F_{n+k}^2 - F_n^4$ meaning F_n^4 is an even integer. However, F_n^4 is even only if F_n is even. □

Proposition 3.2. *F_n is even iff $3|n$.*

Proof. This is an immediate corollary of the fact that $F_m|F_n$ if and only if $m|n$ given that $F_3 = 2$. □

An immediate consequence of these two propositions is the following corollary that restricts the possible values of n for a Fibonacci triangle.

Corollary 3.3. *If (F_{n+k}, F_{n+k}, F_n) is a Fibonacci triangle, then it must be $(F_{3i+k}, F_{3i+k}, F_{3i})$ for some positive integer i.*

Just as we have defined h to be the height of the triangle (F_{n+k}, F_{n+k}, F_n), from now on we will let $d = \gcd(F_{n+k}, F_n)$ for a given triangle (F_{n+k}, F_{n+k}, F_n).

Proposition 3.4. *If (F_{n+k}, F_{n+k}, F_n) is a Fibonacci triangle, then*

$$d = \gcd(F_{n+k}, F_n) = F_{\gcd(n,k)} = \gcd\left(F_{n+k}, \frac{F_n}{2}\right).$$

Proof. That $\gcd(F_{n+k}, F_n) = F_{\gcd(n+k,k)} = F_{\gcd(n,k)}$ follows immediately from the greatest common divisor property of Fibonacci numbers. Let A be the area of the triangle (F_{n+k}, F_{n+k}, F_n). Then $16A^2 = F_n^2(4F_{n+k}^2 - F_n^2)$ and so

$$16A^2 = d^2F_n^2\left(4\frac{F_{n+k}^2}{d^2} - \frac{F_n^2}{d^2}\right).$$

Hence $4\frac{F_{n+k}^2}{d^2} - \frac{F_n^2}{d^2}$ is an integral perfect square which is congruent to $-\frac{F_n^2}{d^2}$ modulo 4. In order for $-\frac{F_n^2}{d^2}$ to be the residue perfect square modulo 4, $\frac{F_n}{d}$ must be even and so $d\,\Big|\,\frac{F_n}{2}$. The immediate consequence is $d\,\big|\gcd\left(F_{n+k}, \frac{F_n}{2}\right)$. Conversely, if $g = \gcd\left(F_{n+k}, \frac{F_n}{2}\right)$ then $g|F_{n+k}$, $g|F_n$, and so $g|d$. □

A result of this proposition is that for (F_{n+k}, F_{n+k}, F_n) a Fibonacci triangle $\frac{F_{n+k}}{d}$, $\frac{F_n}{2d}$, and $\frac{h}{d}$ are all integers that happen to form the sides of a right triangle.

Proposition 3.5. *If (F_{n+k}, F_{n+k}, F_n) is a Fibonacci triangle, then*

$$\left(\frac{F_{n+k}}{d}, \frac{F_n}{2d}, \frac{h}{d}\right)$$

where $d = F_{\gcd(n,k)}$ is a primitive Pythagorean triple.

This means that properties of primitive Pythagorean triples must apply to any Fibonacci triangle as follows.

Corollary 3.6. *If (F_{n+k}, F_{n+k}, F_n) is a Fibonacci triangle, then*

$$\frac{F_{n+k}}{d} \equiv 1 \text{ mod } 4.$$

Corollary 3.7. *Let $d = F_{\gcd(n,k)}$. If (F_{n+k}, F_{n+k}, F_n) is a Fibonacci triangle, then*

$$\frac{F_{n+k}}{d} + \frac{F_n}{2d} \quad \text{and} \quad \frac{F_{n+k}}{d} - \frac{F_n}{2d}$$

must be perfect squares.

Proof. The product $\left(\frac{F_{n+k}}{d} + \frac{F_n}{2d}\right)\left(\frac{F_{n+k}}{d} - \frac{F_n}{2d}\right) = \frac{h^2}{d^2}$ is a perfect square. Additionally $\frac{F_{n+k}}{d} + \frac{F_n}{2d}$ and $\frac{F_{n+k}}{d} - \frac{F_n}{2d}$ have no common factors meaning that for their product to be a perfect square, each must be a perfect square. □

Combining these two corollaries we can get further restrictions on the possible values of n for (F_{n+k}, F_{n+k}, F_n) to be a Fibonacci triangle.

Proposition 3.8. *If (F_{n+k}, F_{n+k}, F_n) is a Fibonacci triangle, then it must be $(F_{6i+k}, F_{6i+k}, F_{6i})$ for some positive integer i.*

Proof. Suppose (F_{n+k}, F_{n+k}, F_n) is a Fibonacci triangle. Then $\frac{F_{n+k}}{d} \equiv 1 \bmod 4$. Consider $\frac{F_{n+k}}{d} + \frac{F_n}{2d} \equiv 1 + \frac{F_n}{2d}$ and $\frac{F_{n+k}}{d} - \frac{F_n}{2d} \equiv 1 - \frac{F_n}{2d}$ modulo 4. If $\frac{F_n}{2d} \equiv 1 \bmod 4$, then $\frac{F_{n+k}}{d} + \frac{F_n}{2d} \equiv 2 \bmod 4$ is a nonresidue and hence not a perfect square. Similarly if $\frac{F_n}{2d} \equiv 2 \bmod 4$, then $\frac{F_{n+k}}{d} + \frac{F_n}{2d} \equiv 3 \bmod 4$ can't be a perfect square. If $\frac{F_n}{2d} \equiv 2 \bmod 4$, then $\frac{F_{n+k}}{d} - \frac{F_n}{2d} \equiv 2 \bmod 4$ can't be a perfect square. Therefore because both $\frac{F_{n+k}}{d} + \frac{F_n}{2d}$ and $\frac{F_{n+k}}{d} - \frac{F_n}{2d}$ must be perfect squares, $\frac{F_n}{2d} \equiv 0 \bmod 4$ implying that $8|F_n$. However $8 = F_6$ divides F_n only if $6|n$. □

The converse to corollary 3.7 is that if either $\frac{F_{n+k}}{d} + \frac{F_n}{2d}$ or $\frac{F_{n+k}}{d} - \frac{F_n}{2d}$ is not a perfect square, then (F_{n+k}, F_{n+k}, F_n) cannot be a Fibonacci triangle. Our strategy is to show through various techniques that either $\frac{F_{n+k}}{d} + \frac{F_n}{2d}$ or $\frac{F_{n+k}}{d} - \frac{F_n}{2d}$ is not a perfect square. A number is not a perfect square if it is a quadratic nonresidue modulo b for some integer b. An example of how this works is the following proposition.

Proposition 3.9. *Let $d = F_{\gcd(n,k)}$. If there exists an odd number b such that the Jacobi symbol*

$$\left(\frac{\frac{F_{n+k}}{d} + \frac{F_n}{2d}}{b}\right) = -1$$

or the Jacobi symbol

$$\left(\frac{\frac{F_{n+k}}{d} - \frac{F_n}{2d}}{b}\right) = -1,$$

then (F_{n+k}, F_{n+k}, F_n) is not a Fibonacci triangle.

Summarizing to this point, the statement "There do not exist any Fibonacci triangles of the form (F_{n+k}, F_{n+k}, F_n) for a fixed value of k," is a consequence of the statement "Given a fixed value k, either $\frac{F_{6i+k}}{d} + \frac{F_{6i}}{2d}$ or $\frac{F_{6i+k}}{d} - \frac{F_{6i}}{2d}$ is not a perfect square for any positive integer i where $d = F_{\gcd(6i,k)}$." As such, we will focus on the latter statement.

4. Tools for Proving the Nonexistence of Fibonacci Triangles

The sequence of Fibonacci numbers modulo n is a repeating sequence where the repeat length depends on the value of n. So for example, modulo 11, the sequence of Fibonacci numbers is

$$\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{8}, \mathbf{2}, \mathbf{10}, \mathbf{1}, \mathbf{0}, 1, 1, 2, 3, 5, 8, 2, \ldots$$

As is easily seen the sequence modulo 11 repeats every tenth number whereas $F_k \equiv F_{k+10t}$ modulo 11. For this reason to compute the values of $F_{6i+k} + \frac{F_{6i}}{2}$ and $F_{6i+k} - \frac{F_{6i}}{2}$ modulo 11, it is enough to compute the values for $1 \le i \le 5$ and $0 \le k \le 10$ and then mod the indices by 10 in order to compute the values of $F_{6i+k} + \frac{F_{6i}}{2}$ and $F_{6i+k} - \frac{F_{6i}}{2}$ modulo 11. In general if l is the repeat length of the Fibonacci numbers modulo n for $n > 2$, then the values of $F_{6i+k} + \frac{F_{6i}}{2}$ and $F_{6i+k} - \frac{F_{6i}}{2}$ modulo n are completely determined by computing the $l^2/2$ values where $1 \le i \le l/2$ and $0 \le k \le l-1$.

An example of this is that the values of $F_{6i+k} + \frac{F_{6i}}{2}$ and $F_{6i+k} - \frac{F_{6i}}{2}$ modulo 11 are completely determined by the matrices

$$a = \begin{bmatrix} 1 & 7 & 4 & 10 & 0 \\ 6 & 8 & 6 & 1 & 1 \\ 3 & 9 & 5 & 4 & 1 \\ 5 & 0 & 6 & 9 & 2 \\ 4 & 3 & 6 & 6 & 3 \\ 5 & 8 & 7 & 8 & 5 \\ 5 & 5 & 8 & 7 & 8 \\ 6 & 7 & 10 & 8 & 2 \\ 7 & 6 & 2 & 8 & 10 \\ 9 & 7 & 7 & 9 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 & 6 & 5 & 7 & 0 \\ 9 & 7 & 7 & 9 & 1 \\ 6 & 8 & 6 & 1 & 1 \\ 8 & 10 & 7 & 6 & 2 \\ 7 & 2 & 7 & 3 & 3 \\ 8 & 7 & 8 & 5 & 5 \\ 8 & 4 & 9 & 4 & 8 \\ 9 & 6 & 0 & 5 & 2 \\ 10 & 5 & 3 & 5 & 10 \\ 1 & 6 & 8 & 6 & 1 \end{bmatrix}$$

where $a_{k+1,i} = F_{6i+k} + \frac{F_{6i}}{2}$ and $b_{k+1,i} = F_{6i+k} - \frac{F_{6i}}{2}$. The nonresidues modulo 11 are $\{2, 6, 7, 8, 10\}$. Through observation of these matrices if $k \equiv 3, 7, 8$ modulo 10, then one of $F_{6i+k} + \frac{F_{6i}}{2}$ and $F_{6i+k} - \frac{F_{6i}}{2}$ is not a perfect square for all i. An example of how this applies is that as a result, in the case $k = 7$, any possible Fibonacci triangle must have $n = 42i$. This then reduces the problem of proving nonexistence in the case of $k = 7$ to proving that either $\frac{1}{F_7}\left(F_{42i+7} + \frac{F_{42i}}{2}\right)$ or $\frac{1}{F_7}\left(F_{42i+7} - \frac{F_{42i}}{2}\right)$ cannot be a perfect square for any value of i.

From this example of using the Fibonacci numbers modulo n we see that it is possible to have reduced the possible cases to the situation where demonstrating the nonexistence of the following term is sufficient for demonstrating nonexistence of Fibonacci triangles for the k value.

Definition 4.1. Let t be the least common multiple of k and 6. Define

$$S_k(i) = \frac{1}{F_k}\left(\frac{F_{ti}}{2} + F_{ti+k}\right).$$

When it is obvious what the value of k is, we will drop the k subscript. Given this definition, the following lemma when combined with proposition 3.9 is sufficient to dealing with $S_k(i)$ cases for i divisible by a large enough power of 2.

Lemma 4.2. *Write $t = 2^v\tau$ where τ is odd. Let $l > v$ be an integer and $g = \gcd(F_k, F_{2^l})$. Then for any odd integer j*

$$S_k(2^{l-v}j) \equiv F_{2^l-1} \mod \frac{F_{2^l}}{g}.$$

Proof. Through application of the identity 2.1, note that

$$F_kS_k(i) = \frac{1}{2}\Big((1+2F_{k-1})F_{ti} + 2F_kF_{ti+1}\Big)$$

and so

$$F_kS_k(2^{l-v}j) = \frac{1}{2}\Big((1+2F_{k-1})F_{t2^{l-v}j} + 2F_kF_{t2^{l-v}j+1}\Big)$$

or

$$F_kS_k(2^{l-v}j) = \frac{1}{2}\Big((1+2F_{k-1})F_{2^l\tau j} + 2F_kF_{2^l\tau j+1}\Big).$$

Then modulo F_{2^l},

$$\frac{1}{2}\Big((1+2F_{k-1})F_{2^l\tau j} + 2F_kF_{2^l\tau j+1}\Big) \equiv F_kF_{2^l-1}.$$

Therefore there exists an integer N such that

$$F_kS_k(2^{l-v}j) = F_kF_{2^l-1} + NF_{2^l}.$$

Since F_k must divide NF_{2^l}, it follows that there exists an integer N' such that $\frac{NF_{2^l}}{F_k} = N'\frac{F_{2^l}}{g}$. Therefore

$$S_k(2^{l-v}j) = F_{2^l-1} + N'\frac{F_{2^l}}{g}$$

and so

$$S_k(2^{l-v}j) \equiv F_{2^l-1} \mod \frac{F_{2^l}}{g}.$$

□

5. The Nonexistence of Fibonacci Triangles for Specific k

In the subsections that follow we will prove the nonexistence of Fibonacci triangles for $6 \leq k \leq 10$. The theorem in the introduction is a direct result.

5.1. The Case When $k = 6$.

We now use these tools to prove the nonexistence of Fibonacci triangles starting with the case $k = 6$. In this case there is one possible value of d when $k = 6$, namely $d = F_6$. We only need to show that $(F_{6i+6}, F_{6i+6}, F_{6i})$ is not a Fibonacci triangle for all positive values of i. As a first step, we have the following proposition:

Proposition 5.1. *If i is odd, then $(F_{6i+6}, F_{6i+6}, F_{6i})$ is not a Fibonacci triangle.*

Proof. Suppose i is odd. By proposition 3.4, if $(F_{6i+6}, F_{6i+6}, F_{6i})$ is a Fibonacci triangle, then $2F_6$ must divide F_{6i}. The Fibonacci numbers modulo 16 are

$$\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{8}, \mathbf{13}, \mathbf{5}, \mathbf{2}, \mathbf{7}, \mathbf{9}, \mathbf{0}, \mathbf{9}, \mathbf{9}, \mathbf{2}, \mathbf{11}, \mathbf{13}, \mathbf{8}, \mathbf{5}, \mathbf{13}, \mathbf{2}, \mathbf{15}, \mathbf{1}, \mathbf{0}, 1, 1, 2, 3, 5, \ldots$$

From observing this sequence, $2F_6 = 16$ divides F_{6i} only if i is even. Therefore $(F_{6i+6}, F_{6i+6}, F_{6i})$ can't be a Fibonacci triangle if i is odd. ☐

Next we consider $S_6(i)$ for i even. For $k = 6$, we have $t = 6$, and so $v = 1$ and $\tau = 3$. So for any even integer i, there exists $l > 1$ and j odd such that $i = 2^{l-1}j$. For all possible values of $l > 1$, $g = \gcd(F_6, F_{2^l}) = F_2 = 1$. So for all even i,

$$S_6(i) \equiv F_{2^l - 1} \mod F_{2^l}$$

where $l > 1$ is chosen such that $i = 2^{l-1}j$ for j odd. However recalling that F_{2^l} is odd, this means that for i even and l chosen as discussed,

$$\left(\frac{S_6(i)}{F_{2^l}}\right) = \left(\frac{F_{2^l-1}}{F_{2^l}}\right) = -1.$$

Therefore if i is even, by proposition 3.9, $(F_{6i+6}, F_{6i+6}, F_{6i})$ is not a Fibonacci triangle. We have now proved the following theorem.

Theorem 5.2. *There are no Fibonacci triangles of the type (F_{n+6}, F_{n+6}, F_n).*

5.2. The Case When $k = 7$.

We start by considering the possible values of d for $k = 7$. This is equivalent to listing the possible values for $\gcd(6i, 7)$ which are $\{1, 7\}$ meaning the possible values for d are $\{F_1, F_7\}$.

Suppose $d = F_1 = 1$. Recall that in section 4, we demonstrated in the example of $F_{6i+k} + \frac{F_{6i}}{2}$ and $F_{6i+k} - \frac{F_{6i}}{2}$ modulo 11 that if $k = 7$, then $F_{6i+k} + \frac{F_{6i}}{2}$ is not a perfect square for all i. Therefore $F_{6i+k} + \frac{F_{6i}}{2}$ is certainly not a perfect square when $\gcd(6i, 7) = 1$. From this we can state that any possible Fibonacci triangle of the form (F_{n+7}, F_{n+7}, F_n) must have $n = 42i$ for some value of i. Note that for $k = 7$, the t value in $S_7(i)$ is 42, and therefore we focus our attention on whether or not $\frac{F_{42i+7}}{F_7} + \frac{F_{42i}}{2F_7} = S_7(i)$ can be a perfect square.

Proposition 5.3. *Suppose $i \not\equiv 0$ modulo 4. Then $S_7(i)$ is not a perfect square.*

Proof. As in the case of computing modulo 11, the sequence of Fibonacci numbers modulo 9 repeats. Therefore it is enough to compute $S_7(i)$ mod 9 which repeats every fourth term and is

$$\mathbf{6}, \mathbf{8}, \mathbf{3}, \mathbf{1}, 6, 8, 3, 1, 6, 8, 3, 1, 6 \ldots$$

In this sequence 6, 8, and 3 are nonresidues modulo 9 and 1 is a residue modulo 9. Since $S_7(i)$ is a nonresidue modulo 9 if $4 \not| i$, $S_7(i)$ can't be a perfect square if $i \not\equiv 0$ modulo 4. □

Theorem 5.4. *There are no Fibonacci triangles of the type (F_{n+7}, F_{n+7}, F_n).*

Proof. We know that if $28 \not| i$, then $\frac{F_{6i+7}}{d} + \frac{F_{6i}}{2d}$ is not a perfect square and therefore $(F_{6i+7}, F_{6i+7}, F_{6i})$ is not a Fibonacci triangle. Consider $S_7(i)$ for $i = 2^{l-1}j$ where $l > 2$ and j is odd. In this situation, by lemma 4.2,

$$S_7(i) \equiv F_{2^l-1} \mod F_{2^l}$$

because $g = \gcd(F_7, F_{2^l}) = 1$. Therefore

$$\left(\frac{S_7(i)}{F_{2^l}}\right) = \left(\frac{F_{2^l-1}}{F_{2^l}}\right) = -1.$$

Applying proposition 3.9 to this last situation removes any possibility that there exists a Fibonacci triangle in the case $k = 7$. □

5.3. **The Case When $k = 8$.** We now consider whether or not there exists any value of i such that $(F_{6i+8}, F_{6i+8}, F_{6i})$ is a Fibonacci triangle. The possible values for d are $\{F_2, F_4, F_8\}$ corresponding to i odd, i congruent to 2 modulo 4, and i congruent to 0 modulo 4. If i is odd, then $d = 1$ and so the fact that one of $F_{6i+8} + \frac{F_{6i}}{2}$ and $F_{6i+8} - \frac{F_{6i}}{2}$ modulo 11 is a nonresidue means that $(F_{6i+8}, F_{6i+8}, F_{6i})$ is not a Fibonacci triangle for i odd.

Suppose i is congruent to 2 modulo 4. Then $d = F_2 = 3$. Note that modulo 11, the inverse of $d = 3$ is 4. So modulo 11 the values of $\frac{1}{F_4}\left(F_{6i+8} + \frac{F_{6i}}{2}\right)$ are completely determined by the repeating sequence

$$\mathbf{6}, \mathbf{2}, \mathbf{8}, \mathbf{10}, \mathbf{7}, 6, 2, 8, 10, 7, \ldots$$

all of which are nonresidues modulo 11. Therefore $\frac{1}{F_4}\left(F_{6i+8} + \frac{F_{6i}}{2}\right)$ is not a perfect square for any value of i and hence $(F_{6i+8}, F_{6i+8}, F_{6i})$ is not a Fibonacci triangle for i congruent to 2 modulo 4.

Finally suppose that i is congruent to 0 modulo 4. Then $d = F_8 = 21$ and it is enough to show that $S_8(m)$ is not a perfect square for all m to show that $(F_{6i+8}, F_{6i+8}, F_{6i})$ is not a Fibonacci triangle for i congruent to 0 modulo 4. First we have the following proposition.

Proposition 5.5. *If m is odd, then $S_8(m)$ is not a perfect square.*

Proof. Computing $S_8(m)$ modulo 23 results in the sequence

$$\mathbf{22}, \mathbf{1}, 22, 1, 22, 1, 22, 1, 22, 1, 22, \ldots$$

The proposition follows from the fact that $-1 \equiv 22$ is a quadratic nonresidue modulo 23. □

Note that in this case $t = 24$ and the v, τ from lemma 4.2 are $v = 3$, and $\tau = 3$. If m is even, there exists $l > 3$ such that $m = 2^{l-3}j$ for j odd such that application of lemma 4.2 implies

$$S_8(m) \equiv F_{2^l-1} \mod \frac{F_{2^l}}{g}$$

where $g = F_8$. Let P be such that $F_{2^l} = 3(7)P = F_8P = gP$. Since $8|2^l$, $F_{2^l-1} \equiv F_7 \mod 3$. This however means $\left(\frac{F_{2^l-1}}{3}\right) = 1$ because $F_7 \equiv 1 \mod 3$. Similarly, the Fibonacci numbers modulo 7 repeat after 16 and therefore $F_{2^l-1} \equiv F_{15} \mod 7$. So $\left(\frac{F_{2^l-1}}{7}\right) = 1$ because $F_{15} \equiv 1 \mod 7$. Applying these computations

$$-1 = \left(\frac{F_{2^l-1}}{F_{2^l}}\right) = \left(\frac{F_{2^l-1}}{3}\right)\left(\frac{F_{2^l-1}}{7}\right)\left(\frac{F_{2^l-1}}{P}\right) = \left(\frac{F_{2^l-1}}{P}\right)$$

because of the property of the Jacobi symbol that says if b_1 and b_2 are odd, then $\left(\frac{a}{b_1b_2}\right) = \left(\frac{a}{b_1}\right)\left(\frac{a}{b_2}\right)$. Therefore

$$\left(\frac{S_8(m)}{P}\right) = \left(\frac{F_{2^l-1}}{P}\right) = -1$$

and so $S_8(m)$ is not a perfect square for m even. It follows that $(F_{6i+8}, F_{6i+8}, F_{6i})$ is not a Fibonacci triangle for i congruent to 0 modulo 4 and we have now proved the following theorem.

Theorem 5.6. *There are no Fibonacci triangles of the type* (F_{n+8}, F_{n+8}, F_n).

5.4. **The Case When $k = 9$.** When $k = 9$, the possible values for d are $\{F_3, F_9\}$ corresponding to $i \not\equiv 0$ modulo 3 and $i \equiv 0$ modulo 3. To see that $\frac{1}{F_3}\left(F_{6i+9} + \frac{F_{6i}}{2}\right)$ cannot be a perfect square if $i \not\equiv 0$ modulo 3, consider $\frac{1}{F_3}\left(F_{6i+9} + \frac{F_{6i}}{2}\right)$ modulo 19. Computing $\frac{1}{F_3}\left(F_{6i+9} + \frac{F_{6i}}{2}\right)$ modulo 19 results in the repeating sequence

$$\mathbf{3}, \mathbf{18}, \mathbf{17}, 3, 18, 17, 3, 18, 17, \ldots$$

Observation that neither 3 nor 18 are quadratic residues modulo 19 means that $\frac{1}{F_3}\left(F_{6i+9} + \frac{F_{6i}}{2}\right)$ cannot be a perfect square if $i \not\equiv 0$ modulo 3.

It is left to check for possible Fibonacci triangles corresponding to $i \equiv 0$ modulo 3. However, if $S_9(m)$ is not a perfect square for all m, then there aren't any Fibonacci triangles corresponding to $i \equiv 0$ modulo 3. Once again we will proceed in two steps. The first step is to consider $S_9(m)$ modulo 8. This is the repeating sequence

$$\mathbf{7}, \mathbf{5}, \mathbf{3}, \mathbf{1}, 7, 5, 3, 1, 7, 5, 3, 1, \ldots$$

which allows us to observe that if $m \not\equiv 0$ modulo 4, then $S_9(m)$ is a nonresidue modulo 8 and hence can't be a perfect square. This leads us to our second step which is the application of lemma 4.2.

If $4|m$, then there exists an integer $l > 1$ such that $m = 2^{l-1}j$ for j odd. Recall that in the case $k = 9$, the values of t, v, and τ are 18, 1, and 9 respectively. Also $g = \gcd(F_9, F_{2^l}) = 1$. So by lemma 4.2

$$S_9(m) \equiv F_{2^l-1} \mod F_{2^l}.$$

Therefore

$$\left(\frac{S_9(m)}{F_{2^l}}\right) = \left(\frac{F_{2^l-1}}{F_{2^l}}\right) = -1$$

and so $S_9(m)$ is not a perfect square when $4|m$. This completes the second step and in turn means there are no Fibonacci triangles corresponding to $i \equiv 0$ modulo 3. In fact,

Theorem 5.7. *There are no Fibonacci triangles of the type* (F_{n+9}, F_{n+9}, F_n).

5.5. **The Case When $k = 10$.** When $k = 10$ there are two possible values of d, namely $F_2 = 1$ and $F_{10} = 55$, depending on whether or not $5|i$ in the triangle $(F_{6i+10}, F_{6i+10}, F_{6i})$. We will consider these two cases in proving the theorem

Theorem 5.8. *There are no Fibonacci triangles of the type* $(F_{n+10}, F_{n+10}, F_n)$.

Proof. If $(F_{n+10}, F_{n+10}, F_n)$ is a Fibonacci triangle, then it is of the form $(F_{6i+10}, F_{6i+10}, F_{6i})$ for some value of i. Modulo 4, $F_{6i+10} + \frac{F_{6i}}{2} \equiv F_{10} \equiv F_4 \equiv 3$ which is a nonresidue modulo 4. Therefore if $5 \not| i$, then $(F_{6i+10}, F_{6i+10}, F_{6i})$ can't be a Fibonacci triangle.

Recall

$$S_{10}(m) = \frac{1}{F_{10}}\left(\frac{F_{tm}}{2} + F_{tm+10}\right)$$

where $t = 30 = 2^1(15)$. When $5|i$, $(F_{6i+10}, F_{6i+10}, F_{6i})$ is a Fibonacci triangle only of $S_{10}(i/5)$ is a perfect square. First we consider the sequence $S_{10}(m)$ modulo 161 which is

$$\mathbf{102}, \mathbf{95}, \mathbf{120}, \mathbf{160}, \mathbf{59}, \mathbf{66}, \mathbf{41}, \mathbf{1}, 102, 95, 120, 160, 59, 66, 41, 1, \ldots$$

The odd entries in this sequence which repeats every 8 terms are $\{102, 120, 59, 41\}$, all of which are nonresidues modulo 161. Therefore if m is odd, $S_{10}(m)$ is not a prefect square.

Next note that $g = \gcd(F_{10}, F_{2^l}) = F_2 = 1$ for any integer l. Suppose m is even. Then there exists integers l and j for $l > 1$ and j odd such that $m = 2^l j$. By lemma 4.2

$$S_{10}(m) \equiv F_{2^l-1} \mod F_{2^l}$$

meaning

$$\left(\frac{S_{10}(m)}{F_{2^l}}\right) = \left(\frac{F_{2^l-1}}{F_{2^l}}\right) = -1.$$

Therefore, if m is even, $S_{10}(m)$ is also not a perfect square. Hence there are no Fibonacci triangles of the type $(F_{6i+10}, F_{6i+10}, F_{6i})$. □

References

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